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A THEORY OF THE PROPAGATION OF ELECTROMAGNETIC WAVES IN A MAGNETICALLY ACTIVE MEDIUM

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The article is devoted to development of a theory of the propagation of electromagnetic waves in a magnetically active medium (in the ionosphere with consideration of the effect of the earth's magnetic field). An approximate solution is performed for the equations of propagation in the region of reflection of a wave from a layer and at the beginning of a layer, where ~~whose~~ geometrical optics is not applicable. The effect of "tripling" of signals is considered in an approximation which holds for small angles between the magnetic field and the direction of propagation. The problem of the effective ~~active~~ field is clarified, and by means of the ~~a~~ kinetic-equation method an expression is obtained for the effective number of collisions, taking into consideration the presence of the magnetic field.

1. A general consideration of the problem of the propagation of electromagnetic waves in a magnetically active medium (in the ionosphere with consideration of the effect of the earth's magnetic field) has been made by us earlier (1) (reference (1) is henceforth cited as I). With normal incidence of a wave on a layer whose properties depend only on height (coordinate x), the electromagnetic field of the wave is subject to the equations (See I, (14)-(16):

$$(d^2E_y/dx^2) + (\omega^2/c^2)(AE_y + iCE_z) = 0$$
$$(d^2E_z/dx^2) + (\omega^2/c^2)(-iCE_y + BE_z) = 0, \tag{1}$$

where

$$A = \frac{u-(1-v)^2 - uv\cos^2\alpha}{u-(1-v)-uv\cos^2\alpha}, \quad B = \frac{u(1-v)-(1-v)^2}{u-(1-v)-uv\cos^2\alpha}, \quad (2)$$

$$C = \frac{u^{1/2}\cos\alpha\cdot v(1-v)}{u-(1-v)-uv\cos^2\alpha}$$

$$u = \omega_H^2/\omega^2; \quad \omega_H = eH/mc, \quad v = \omega_0^2/\omega^2 = 4\pi e^2 N(x)/m\omega^2$$

The earth's magnetic field $H^{(0)}$ is assumed to lie in the plane xz and to form with the direction of propagation -- the x axis -- the angle α. In (1)-(2) ω is the ~~cyclical~~ angular velocity of the wave and N(x) is the concentration of electrons in the layer.

In the approximate geometrical optics considered in I, the solution of equation (1) is as follows:

$$E^{(0)}_{y;1,2} = E^{(0,0)}_{y;1,2} \exp\left\{\pm(i\omega/c)\int n_{1,2}\,dx\right\} = \quad (3)$$

$$= \left(\mathrm{const}/\sqrt{n_{1,2}(1-R^2_{1,2})}\right)\exp\left\{\pm(i\omega/c)\int n_{1,2}\,dx\right\},$$

$$E^{(0)}_{z;1,2} = R_{1,2}\,E^{(0)}_{y;1,2},$$

where $n^2_{1,2}$ is the solution of the equation

$$\begin{vmatrix} A-n^2 & iC \\ -iC & B-n^2 \end{vmatrix} = 0,$$

and thus

$$n^2_{1,2} = 1 - \frac{2v(1-v)}{2(1-v) - u\sin^2\alpha \pm \sqrt{u^2\sin^4\alpha + 4u(1-v)^2\cos^2\alpha}} \quad (4.1)$$

Further

$$R_{1,2} = \frac{iC}{B-n^2_{1,2}} = \frac{A-n^2_{1,2}}{-iC} =$$

$$= -i\,\frac{2u^{\frac{1}{2}}(1-v)\cos\alpha}{u\sin^2\alpha \mp \sqrt{u^2\sin^4\alpha + 4u(1-v)^2\cos^2\alpha}} \quad (4.2)$$

The indexes 1 and 2, which correspond to the upper and lower signs in front of the root in (4.1) and (4.2), characterize the type of wave, i.e., an "~~conventional~~ ordinary" (index 2) or an "~~unconventional~~ extraordinary" (index 1) wave. Solution (3) rigorously satisfies the equations:

$$(A-n^2)E_y^{(0)} + iCE_z^{(0)} = 0; \quad -iCE_y^{(0)} + (B-n^2)E_z^{(0)} = 0;$$

$$\frac{dE_y^{(0,0)}}{dx} + \left(\frac{dn/dx}{2n} - \frac{R\,dR/dx}{1-R^2}\right)E_y^{(0,0)} = 0;$$

$$\frac{dE_z^{(0,0)}}{dx} + \left(\frac{dn/dx}{2n} - \frac{dR/dx}{R(1-R^2)}\right)E_z^{(0,0)} = 0 \qquad (5)$$

Without giving more detailed attention to the conditions of applicability of the approximation of geometrical optics (see I), we shall note here only that in the case of a smooth ionosphere layer(1,2), this approximation is inapplicable only when the inequality $[(\lambda/2\pi) = (c/\omega)]$ is not observed:

$$(\lambda/2\pi)\left|(dn_{1,2}/dx)/n_{1,2}^2\right| \ll 1 \qquad (6)$$

and when $v \to 0$, i.e., at the very beginning of the layer.

The inequality (6) is not observed in the region of reflection of the waves from the layer, where the index of refraction $n_{1,2}$ is small and, in particular, when it is equal to zero. Ordinarily in such cases non-observance of the condition of applicability of geometrical optics [condition (6)] to one of the waves ("~~conventional~~ ordinary" or "~~unconventional~~ extraordinary") ~~involves violation of~~ does not mean that this condition ~~for~~ does not apply to the wave of the other type. However, when $\alpha \to 0$, geometrical optics is simultaneously inapplicable to waves of both types, and the "tripling" effect occurs (see I).

In cases where geometrical optics is not applicable it is necessary to find a rigorous or correct approximate solution of system (1), which hitherto has not been done and to which the present article is essentially devoted. In Paragraphs 2 and 3 investigation is made of the solution in the region of reflection of a wave from a layer. In Paragraph 4 brief consideration is given to the solution at the beginning of the layer (the region $v \to 0$). Paragraph 5 is devoted to the "tripling" effect. In Paragraph 6 the discussion involves problems of the ~~active~~ effective field and the effective number of collisions in the ionosphere in the presence of a magnetic field.

2. In the absence of a magnetic field, when $u = 0$, equation (1) assumes the form:

$$(d^2 E_{y,z}/dx^2) + (\omega^2/c^2) n_0^2 E_{y,z} = 0,$$

$$n_{1,2}^2 = n_0^2 = 1 - v = 1 - (4\pi e^2 N(x)/m\omega^2). \qquad (7)$$

With the exception of frequencies very close to the critical frequency, the layer in the region of reflection, where geometrical optics is inapplicable, may be replaced by a linear form [2,3], i.e., it may be assumed that $v = ax$. In this case the change in phase of the wave as the result of its passage into the layer and reflection from it is equal to [2,4]:

$$\varphi = \frac{2\omega}{c} \int_0^{x(n=0)} n\,dx - \frac{\pi}{2}. \qquad (8)$$

The solution of equation (7) for a linear layer is expressed by the ~~Eyri~~ Airy integral or in Bessel functions of the order ±1/3 [2-4].

In the presence of a magnetic field the equations of (1) reduce to an equation of the type of (7) in two particular cases -- with longitudinal ($\alpha = 0$) and transverse ($\alpha = \pi/2$) propagation -- when

$$\alpha = 0: n_{1,2}^2 = A \pm C = 1 - v(1 \mp u^{\frac{1}{2}})^{-1}, \; A = B = (u - 1 + v)/(u-1),$$

$$C = u^{\frac{1}{2}} v/(u-1), \; R_{1,2} = \pm i; \qquad (9)$$

$$\alpha = \pi/2 : n_1^2 = A = 1 - [v(1-v)/(1-u-v)],\ C=0,\ R_1 = 0, \qquad (10)$$
$$n_2^2 = n_0^2 = B = 1 - v,\ R_2 = i\infty$$

In the case of (9) with the variables $F_\pm = E_y \pm iE_z$, the system ~~of~~ (1) assumes the form:

$$\frac{d^2F_\pm}{dx^2} + \frac{\omega^2}{c^2}(A \pm C)F_\pm = \frac{d^2F_\pm}{dx^2} + \frac{\omega^2}{c^2}\left(1 - \frac{v}{1 \mp u^{1/2}}\right)F_\pm = 0. \qquad (11)$$

With transverse propagation [the case of (10)] we have:

$$\frac{d^2E_y}{dx^2} + \frac{\omega^2}{c^2}\left(1 - \frac{v(1-v)}{1-u-v}\right)E_y = 0,$$
$$\frac{d^2E_z}{dx^2} + \frac{\omega^2}{c^2}(1-v)E_z = 0. \qquad (12)$$

We see that in both cases the variables are separable and the system of (1) actually reduces to two equations of type (7). Hence it follows that with $\alpha = 0$ and $\alpha = \pi/2$ for the phase φ, and also for the field itself, there occur results which are correct in the absence of a field, of course, with the substitution of $n_{1,2}$ for n_0 (i.e., for example, for φ formula (8) is correct with $n = n_{1,2}$). For the equations of (11) and the second of the equations of (12) this assertion is obvious, since they are identical with (7) in the sense that v enters into them linearly. Into the first of the equations of (12) v enters in more complex fashion, and therefore, even for a linear layer

$$v = ax + b \qquad (13)$$

and n_1^2 is a non-linear function of x. However, for correctness of formula (8) and of the entire solution for the linear layer it is merely necessary that n_1^2 may be replaced by a linear expression only in the region where geometrical optics is inapplicable (2-4). Selecting the origin of the coordinates at the point $n_1^2 = 0$ (assuming that

$u < 1$), we let:

$$v = 1 - u^{\frac{1}{2}} + ax. \tag{14}$$

Then, on condition that

$$|ax/(1 - u)| \ll 1, \tag{15}$$

$$n_1^2 \cong -2ax/(1 - u^{\frac{1}{2}}), \tag{16}$$

and the first equation of (12) reduces to (7) with n_0^2 as a linear function of x.

Application of the condition of (6) to (16) shows that geometrical optics is applicable if

$$x \gg \left(\frac{1 - u^{1/2}}{8a} \cdot \frac{\lambda^2}{4\pi^2}\right)^{1/2} \tag{17}$$

Thus, if the conditions of (15) and (17) are simultaneously fulfilled, then formula (8) and the others are correct also as applied to the first equation of (12). For the F-layer, for example, $a \sim 10^{-6} - 10^{-7}$, $\lambda \sim 6 \cdot 10^3$ and when $1 - u^{\frac{1}{2}} \sim 1$, condition (15) denotes that $x \ll 10^6 - 10^7$, and condition (17) that $x \gg 10^4$, i.e., both inequalities are completely compatible.

3. The breaking down of the equations of (1) into two independent equations of the second order takes place only in the two indicated instances, $\alpha = 0$ and $\alpha = \pi/2$, typified by the fact that in their case the wave-shape is not a function of x (i.e., R = const). In general then, R = R(x), the equations do not break down, and may be reduced only to one equation of the fourth order for E_y or E_z, i.e., they are extremely complex. Therefore, it is expedient here to resort

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to an approximate solution. With the exception of the "tripling" region, which lies for small values of α in the region of reflection of the waves from the layer, geometrical optics is inapplicable only for a wave of the one type, and is applicable for a wave of the second type. This obviously ties in with the fact that a wave of type 2 is reflected near the point $n_2^2 = 0$ (i.e., $v_{20} = 1$), and a wave of type 1 is reflected at one of the points $n_1^2 = 0$ (i.e., $v_{10\pm} = 1 \pm u^{\frac{1}{2}}$). Therefore, in the region of reflection the waves of both types are independent to a good approximation (see I for more details), and it may be expected that each of the waves is approximately subject to an equation of type (1) with $n_{1,2}^2$ substituted for n_0^2 just as takes place in the cases of (9) and (10). This assumption proves to be correct.

If we ~~go over~~ change to the variables

$$F_\pm = E_z \pm RE_y, \tag{18}$$

then the equations of (1) assume the form ($R' \equiv dR/dx$, $R'' \equiv d^2R/dx^2$):

$$\frac{d^2F_\pm}{dx^2} - \frac{R'}{R}\frac{dF_\pm}{dx} + \frac{\omega^2}{c^2}\left\{\frac{A+B}{2} \pm i\frac{C^2(R^2-1)}{2R}\right\}F_\pm + \left\{\left(\frac{R'}{R}\right)^2 - \frac{R''}{2R}\right\}F_\pm +$$
$$+ \frac{R'}{R}\frac{dF_\pm}{dx} + \frac{\omega^2}{c^2}\left\{\frac{B-A}{2} \pm \frac{iC(R^2+1)}{2R}\right\}F_\pm - \left\{\left(\frac{R'}{R}\right)^2 - \frac{R''}{2R}\right\}F_\mp = 0 \tag{19}$$

Taking into consideration that by virtue of (2) and (4)

$$\frac{A+B}{2} + \frac{iC}{2R}(R^2 - 1) - n^2 = \frac{B-A}{2} - \frac{iC(R^2+1)}{2R} = 0, \tag{20}$$

and evaluating the order of the individual ~~members~~ terms of the equation (19), it is natural to seek a solution of this system in the form:

$$F_\pm = F_\pm^{(0)} + F_\pm^{(1)}, \; |F_\pm^{(1)}| \ll |F_\pm^{(0)}| \;, \qquad (21)$$

and the values of $F_\pm^{(0)}$ satisfy the equations:

$$\frac{d^2F_+^{(0)}}{dx^2} - \frac{R'}{R}\frac{dF_+^{(0)}}{dx} + \frac{\omega^2}{c^2}n^2F_+^{(0)} + i\frac{\omega^2}{c^2}C\frac{R^2+1}{R}F_-^{(0)} = 0, \qquad (22)$$

$$\frac{R'}{R}\frac{dF_+^{(0)}}{dx} - i\frac{\omega^2}{c^2}C\frac{R^2-1}{R}F_-^{(0)} = 0.$$

It is easy to ~~be convinced of the fact~~ see that in the approximation of geometrical optics the solutions of the systems of (19) and (20) coincide, as indeed must be the case, with the results given in Paragraph 1 (in this approximation $F_+ = 2E_z = 2RE_y$ and $F_- = 0$).

Making use of (20) - (22), we obtain from (19) the equation for $F_\pm^{(1)}$; by eliminating ~~excluding~~ in these equations and in the equations ~~of~~ (22) $F_-^{(1)}$ and $F_-^{(0)}$ respectively ~~correspondingly~~, after simple transformations, we obtain

$$\frac{d^2F^{(0)}}{dx^2} + \frac{2R'}{R(R^2-1)}\frac{dF_+^{(0)}}{dx} + \frac{\omega^2}{c^2}n^2F_+^{(0)} = 0, \qquad (23)$$

$$\frac{d^2F_+^{(1)}}{dx^2} \; \frac{2R'}{R(R^2-1)}\frac{dF_+^{(1)}}{dx} + \frac{\omega^2}{c^2}n^2F_+^{(1)} = -\frac{2}{R^2+1}\left\{\frac{R''}{2R} - \left(\frac{R'}{R}\right)^2\right\}(F_+^{(0)} - F_-^{(0)}) +$$

$$+ \frac{2R'}{R(R^2-1)}\frac{dF_-^{(0)}}{dx} - \frac{\omega^2}{c^2}\left(\frac{R^2+1}{R^2-1}\right)F_-^{(0)} - \frac{R^2+1}{R^2-1}\frac{d^2F_-^{(0)}}{dx^2} \equiv f(x). \qquad (24)$$

If the condition ~~of~~ (21) is fulfilled, then the approximate solution of the problem has the form $F_\pm^{(0)}$ and is obtained from the equations of (22) or (23).

In order to proceed further, the behavior of the functions $R_{1,2}$ near the points of reflection $v_{20} = 1$ and $v_{10\pm} = 1 \pm u^{\frac{1}{2}}$ is essential (R_1 corresponds to ~~selection of~~ the lower sign for the root in (4) and R_2 to ~~selection of~~ the upper sign; we note also that

$R_2 = 1/R_1$). With precision up to terms of the first order in

$(v - v_{20})$ and $(v - v_{10\pm})$ we have:

$$R_1(v \cong 1 \pm u^{1/2}) = \pm i \cos\alpha + [(i \sin^2\alpha \cos\alpha)/u^{\frac{1}{2}}(1+\cos^2\alpha)](v - 1 \mp u^{\frac{1}{2}});$$

$$R_2(v \cong 1) = i\frac{u^{1/2}\sin^2\alpha}{(1-v)\cos\alpha}; \left(\frac{dR_1}{dx}\right)_{v_{10\pm}} = \frac{i \sin^2\alpha \cos\alpha}{u^{1/2}(1+\cos^3\alpha)}\left(\frac{dv}{dx}\right)_{v_{10\pm}}; \quad (25)$$

$$\left(\frac{d^2R_1}{dx^2}\right)_{v_{10\pm}} = \frac{i \sin^2\alpha\cos\alpha}{u^{1/2}(1+\cos\alpha)}\left(\frac{d^2v}{dx^2}\right)_{v_{10\pm}} + i\frac{2\sin^2\alpha\cos\alpha(3+3\cos^2\alpha+\cos^4\alpha)}{u(1+\cos^2\alpha)^3}\left(\frac{dv}{dx}\right)_{v_{10\pm}}$$

Let us consider the propagation of a wave of type I in the region of points $v_{10\pm} = 1 \pm n^{\frac{1}{2}}$; in this case we must let everywhere $n^2 = n_1^2$ and $R = R_1$ in (19)-(24) (for a wave of type 2 it is necessary to let $n^2 = n_2^2$ and $R = R_2$ in these equations, but this case lacks interest, since for wave 2 here, if the angle α is not too small, the approximation of geometrical optics is applicable).

~~Putting~~ in (23), setting

$$F_+^{(0)} = G_+^{(0)} R_1/\sqrt{1 - R_1^2} \quad (26)$$

for $G_+^{(0)}$ we obtain:

$$\frac{d^2G_+^{(0)}}{dx^2} + \left\{\frac{\omega^2}{c^2}n_1^2 + \frac{(3R_1^2-2)(R_1')^2}{R_1^2(R_1^2-1)^2} - \frac{R_1''}{R_1(R_1^2-1)}\right\}G_+^{(0)} = 0 \quad (27)$$

This is an equation of type (7) and differs from the latter by simple replacement of n_o^2 by n_1^2 and in members containing R. These members are of the order $(dv/dx)^2_{v_{10}}$ and $(d^2v/dx^2)_{v_{10}}$, i.e., under normal conditions $\backsim 10^{-12}$-10^{-14} ($v \backsim ax \backsim 10^{-6}$-$10^{-5}x$; see Paragraph 2). Simultaneously $\omega^2/c^2 \backsim 10^{-6}$, and consequently the members with R are ~~essential~~ important only when $n_1^2 \lesssim 10^{-5}$-10^{-7}, while to the first maximum of the field intensity corresponds the value (see (2,3)):

$$n_1^2 \cong \frac{dn_1^2}{dx}\left(\frac{\lambda^2}{4\pi^2}\frac{1}{dn_1^2/dx}\right)^{1/3} = \left(\frac{dn_1^2}{dx}\,\frac{\lambda}{2\pi}\right)^{2/3} \sim 10^{-2}.$$

Thus the effect of Terms containing ~~members with~~ R is infinitesimal, and consists only ~~it reduces~~ of a shift ~~to a displacement~~ of the point Δx ~~current~~ ($\Delta x \backsim 0.1$-1), where the coefficient of $G_+^{(0)}$ changes to zero (we assume for purposes of approximation ~~orientation~~ that $n_1^2 \backsim ax$), and to an infinitesimal change $\Delta\varphi \backsim 10^{-6}$ in the phase φ. Thus in the neighborhood of the point of reflection the solution of the equations of propagation reduces to equation (7) with a natural substitution of n_1^2 for n_0^2 and appearance of the amplitude factor $R/\sqrt{1 - R^2}$ (see (26)), whose presence is ~~still~~ evident even in the approximation of geometrical optics (see (3)). It is, of course, true that this conclusion is valid only if condition (21) is observed i.e., if the values of $F_\pm^{(0)}$ are actually an approximate solution for the system ~~of~~ (19).

The solution of equation (24) has the form:

$$F_+^{(1)} = F_0 \int \left\{ \frac{R^2}{F_0^2(R^2-1)} \int \frac{f(x)\,F_0}{R^2}(R^2-1)\,dx \right\} dx \tag{28}$$

where F_0 is the solution of the same homogeneous equation. According to (22):

$$F_-^{(0)} = (c^2/\omega^2)(R'/iC(R^2 - 1))(dF_+^{(0)}/dx); \tag{29}$$

in the neighborhood of the points $v_{10\pm} = 1 \pm u^{\frac{1}{2}}$, C does not approach zero and R_1 approaches ~~approached~~ neither zero nor infinity (see (25)). In this case the main term ~~member~~ in $f(x)$ is less than

$$(\omega^2/c^2)F_-^{(0)} \backsim (dv/dx)_{v_{10\pm}}\, dF_+^{(0)}/dx \lesssim (dv/dx)_{v_{10\pm}}(2\pi/\lambda)F_+^{(0)}, \tag{30}$$

where for simplicity constants ~~multipliers~~ containing u and $\sin^2 \alpha$, which are, generally ~~speaking~~, of the order of unity, have been omitted.

As a matter of fact, in the approximation of geometrical op-

optics $(d^2F_-^{(0)}/dx^2) - (\omega^2/c^2)n_1^2F_-^{(0)} \cong (\omega/c)n_1^2F_-^{(0)}$, and in the region $n_1^2 \cong 0$ the deviation from the approximation of geometrical optics is of the order of the field magnitude itself, i.e.,

$$(d^2F_-^{\circ}/dx^2) - (\omega^2/c^2)n_1^2F_-^{\circ} \sim (\omega^2/c^2)n_1^2(x \sim \lambda/2\pi)F_-^{(0)} < (\omega^2/c^2)F_-^{(0)}$$

(see the solutions in (2-4); further, $dF^{(0)}/dx \sim (2\pi/\lambda)n_1F_+^{(0)} < < (2\pi/\lambda)F_+^{(0)}$, whence expression (30) is derived. Substituting ~~Putting~~ the value of f(x) from (30) in~~to~~ (28), it may be seen that

$$F_+^{(1)} < (dv/dx)_{v_{10\pm}} xF_+^{(0)}. \qquad (31)$$

In the region (which is interesting to us) around the point $n_1^2 = \int(dv/dx)x \sim ax \ll 1$, and thus the solution of $F_+^{(0)}$ is a good approximation, the more so in that actually in (31) the sign $\ll$ may be substituted for the sign $<$.

For a wave of type 2 in the neighborhood of the point of reflection $v_{20} = 1$, where $n_2^2 = 0$, on the other hand, the approximation which has been made is not valid. Actually, as follows from (2) and (25), in the neighborhood of this point $C \sim (1 - v)$, and $R_2 \sim 1/(1 - v)$, in (29) $F_+^{(0)} \sim (1 - v)^{-1}dF_+^{(0)}/dx$ and in f(x) the principal ~~member~~ Term $\sim (1 - v)^{-3}F_+^{(0)}$. Therefore, condition (21) is not fulfilled and the entire approximation is invalid, and therefore special attention must be given to the region $v \cong 1$.

In this region we shall let

$$v = 1 + ax, \quad ax \ll 1. \qquad (32)$$

Then with a precision extending to infinitesmals of higher order ~~members of higher order of smallness~~ the system ~~of~~ (1) assumes the form:

$$\frac{d^2E_y}{dx^2} + \frac{\omega^2}{c^2}\left\{\left(1 - \frac{ax}{u\sin^2\alpha}\right)E_y - i\,\frac{\cos\alpha\cdot ax}{u^{1/2}\sin^2\alpha}E_z\right\} = 0, \quad (33)$$

$$\frac{d^2E_z}{dx^2} + \frac{\omega^2}{c^2}\left\{i\,\frac{\cos\alpha\cdot ax}{u^{1/2}\sin^2\alpha}E_y - \frac{ax}{\sin^2\alpha}E_z\right\} = 0 \quad (34)$$

Further,

$$n_2^2 = -ax/\sin^2\alpha \quad ; \quad R_2 = -iu^{\frac{1}{2}}\sin^2\alpha\,/ax\cos\alpha\,. \quad (35)$$

With the approximation of geometrical optics and on condition that

$$|1/R_2| = |ax\cos\alpha\,/u^{\frac{1}{2}}\sin^2\alpha| \ll 1, \quad (36)$$

$$E_z^{(0)}/E_y^{(0)} = R_2 \gg 1.$$

This fact suggests the use of the approximation

$$E_z = E_z^{(0)} + E_z^{(1)}; \quad E_y = E_y^{(1)}, \quad (37)$$

$$\left|E_z^{(1)}\right| \ll \left|E_z^{(0)}\right|, \quad \left|E_z^{(1)}\right| \ll \left|E_z^{(0)}\right|$$

where

$$\frac{d^2E_z^{(0)}}{dx^2} - \frac{\omega^2 ax}{c^2\sin^2\alpha}E_z^{(0)} = \frac{d^2E_z^{(0)}}{dx^2} + \frac{\omega^2}{c^2}n_2^2E_z^{(0)} = 0. \quad (38)$$

For $E_{y,z}^{(1)}$ we have:

$$\frac{d^2E_y^{(1)}}{dx^2} + \frac{\omega^2}{c^2}\left(1 - \frac{ax}{u\sin^2\alpha}\right)E_y^{(1)} = i\,\frac{\omega^2}{c^2}\frac{\cos\alpha}{u^{1/2}\sin^2\alpha}axE_z^{(0)} = f(x) \quad (39)$$

$$\frac{d^2E_z^{(1)}}{dx^2} + i\,\frac{\omega^2}{c^2}\frac{\cos\alpha}{u^{1/2}\sin^2\alpha}axE_y^{(1)} - \frac{\omega^2 ax}{c^2\sin^2\alpha}E_z^{(1)} = 0. \quad (40)$$

The solution of equation (39) is as follows:

$$E_y^{(1)} = E_{y0}\int\left\{E_{y0}^{-1}\int f(x)E_{y0}\,dx\right\}dx \qquad (41)$$

where E_{y0} is the solution of the homogeneous equation (39). For evaluation of the magnitude of (41) we may let $E_{y0} = e^{\pm j\omega x/c}$; taking into consideration further that $n_2^2 \ll 1$ and that therefore the dependency of $E_z^{(0)}$ on x is considerably less pronounced than for E_{y0}, we have:

$$E_y^{(1)} \sim i(a\cdot\cos\alpha/u^{\frac{1}{2}}\sin^2\alpha)(x + 2j(c/\omega))E_z^{(0)}. \qquad (42)$$

With the condition of (36) and, strictly speaking, if this condition is valid also for $x \sim \lambda$, the second inequality of (37) is derived from (42); the first of these inequalities follows from the second and from equation (40). Thus, with the conditions of (32) and (36) satisfied, the approximation [redacted] used ~~adopted~~ is valid. Let us note that in the case being analyzed the applicability of the approximation of geometrical optics, as follows from (6) and (35), is equivalent to observance of the inequality;

$$x \gg \left[(\lambda/4\pi^2)(\sin^2\alpha/a)\right]^{1/3}. \qquad (43)$$

For example, when $a \sim 10^{-6}$, $\lambda/2\pi \sim 10^3$ and $u^{\frac{1}{2}} \sim \sin\alpha \sim \cos\alpha \sim 1$, the conditions of (32) and (36) denote that $x \ll 10^6$, and the condition of (43) denotes that $x \gg 10^4$. In this instance the approximation which has been selected is valid up to the region where geometrical optics become applicable. For small values of α the situation gets worse, but even when $\sin\alpha \sim \alpha \sim 1/10$, the conditions of (36) and (43) are as follows: $x \gg 10^4$ and $x \gg 2\cdot 10^3$. With values of α as small and smaller than this our approximation cannot be valid due to the "tripling" effect (see Paragraph 5).

The equation ~~of~~ (38) has the form of equation (7) with n_2^2 substituted for n_0^2, and thus the formula of (8) and the other results obtain~~able~~ed for the isotropic problem ~~are carried over~~ can be extended to the solution being considered.

When the frequency nears the critical value, ~~filtering of~~ the waves (begin to penetrate) through the layer ~~begins~~; as has been pointed out in [2,3], if the ~~coefficient of~~ transmi~~ssion~~ttance $D \ll 1$, then

$$D = \exp\left\{-(2\omega/c)\int_{x_1}^{x_2} |n|\, dx\right\}, \qquad (44)$$

where the integration is performed between the points x_1 and x_2, at which $n^2 = 0$.

We note that in [2,3] the formula (44) is given as approximate, with an ~~precision up to the pre-exponential multiplier~~ accuracy equal to that of the constant in front of the exponential; it may, however, be shown that (with an ~~precision~~ accuracy) up to members of a higher order of smallness (the order D^2), ~~that~~ formula (44) is ~~precise~~ exact. Without giving more detailed attention to this problem, we note that in the presence of a magnetic field the expression (44) is equally valid, it being true of course that $|n| = |n_{1,2}|$.

~~Let it~~ We also ~~be~~ note~~d~~ that generalization of all the results obtained for the case where there is absorption, which has not been taken into consideration above, may be performed without difficulty just as for an isotropic medium [3].

4. Geometrical optics is also inapplicable at the beginning of the layer, where $v \to 0$. In this region the polarization of both waves, which is obtained according to (4), is generally speaking entirely different (see below); simultaneously, with very slight anisotropy, propagation of the waves should approximate ~~approach in nature~~ propagation in an isotropic medium, which does hold ~~not occur~~ in the approximation of geometrical optics (for more details, see (5)). The condition of applicability of geometrical optics in a medium with slight anisotropy is as follows*: where a is a parameter which characterizes the properties of the medium as a function of x (for example, the coefficient in the expression $v = ax$), and Δn is the difference in indices of refraction of waves of different types.

With a small value of v, with a precision up to members of the first order, we have:

$$A = 1 + [v/(u - 1)], \quad B = 1 + [v(1 - u \sin^2 \alpha)/(u - 1)],$$

$$C = [u^{\frac{1}{2}} \cos \alpha \cdot v/(u - 1)];$$

$$n^2_{1,2} = 1 - \left\{ 2v/[2 - u \sin^2 \alpha \pm (u^2 \sin^4 \alpha + 4u \cos^2 \alpha)^{\frac{1}{2}}] \right. ; \qquad (46)$$

$$\Delta n = n_2 - n_1 = \frac{(u^2 \sin^4 \alpha + 4u \cos^2 \alpha)^{\frac{1}{2}}}{1 - u} v;$$

$$R_{1,2}(v = 0) = -i \frac{2 \sqrt{u} \cos \alpha}{u \sin^2 \alpha \mp (u^2 \sin^4 \alpha + 4u \cos^2 \alpha)^{\frac{1}{2}}}$$

Letting $v = ax$, where x is measured from the beginning of the layer, we obtain from (45) and (46) the condition of applicability of geometrical optics in the form

$* \ a\lambda / 2\pi \Delta n \ll 1 \qquad (45)$

$$\lambda/2\pi x \ll 1 \qquad (47)$$

where for simplicity it is assumed that $u - 1 \sim 1$.

The condition (47) is also obtained from one of the criteria of the applicability of geometrical optics given in I (see I (33.3)), specifically, from the condition

$$\frac{\lambda}{2\pi}\left(\frac{R}{C}\right)\frac{nn'RR'}{(1 - R^2)(n)(1 - R^2)'} \ll 1.$$

It is easy to see that the inequality (47) is retained also for a function of the type $v = ax^2$ or $v = ax^3$. The formulas (46) are valid if

$$v \ll |(u - 1)/(u \cos^2\alpha - 1)| \; ; \; v \ll 1 \qquad (48)$$

In this case we proceed from equation (1), using the values of (46) as A, B, and C. An approximate solution of this system may be ~~achieved~~ obtained if we take as the zero approximation either a solution of the type $E^{(0)}_{y,z} = \text{const}\cdot e^{\pm i\omega x/c}$, or a solution based on the approximation of geometrical optics. Passing over the calculations, we shall only indicate their result for the case where $v = ax$ (for a function of the type $v = ax^k$, where k is not too large, the same result is obtained). Corrections ~~to~~ for the zero ~~order~~ approximation, i.e., the ratio of ~~members~~ Terms of the first approximation to the zero one, are of the order of ax and $2\pi ax^2/\lambda$, (The latter predominating in the region where geometrical optics is applicable.) ~~which, however~~, under the conditions of the ionosphere, ~~are~~ This term is much less than unity. Actually, when $a \sim 10^{-6} - 10^{-7}$ and $\lambda/2\pi \sim 10^3$, $2\pi ax^2/\lambda \sim 10^{-9} - 10^{-10}x^2$; i.e., when $x \sim 10^4$, $2\pi ax^2/\lambda \sim 0.1 - 0.01 \ll 1$ and $\lambda/2\pi x \sim 0.1 \ll 1$. From this it follows that the correction to the phase and the ratio of amplitudes E_z/E_y is much less than unity and thus the phase and, most important, the polarization of the waves reflected from the ionosphere (the F-layer is ~~in question~~ meant), are obtained to a good approximation by application of geometrical optics from the very beginning of the layer without consideration of the region where the condition (47) is not fulfilled and geometrical optics is inapplicable. This result is ~~obtained~~ explained by the fact that we are interested in the phase which is essentially determined by the value $n^2 = 1$; the difference in phases, determined by the value $\Delta n \ll 1$, is obtained incorrectly from the approximation of geometrical optics, but this difference itself is of the order of $2\pi ax^2/\lambda \ll 1$ and is not essential to our problem.

5. With small values of the angle α there should occur the "tripling" effect, mentioned and qualitatively discussed in I and consisting of partial reflection of a wave of type 2 from the point $v_{20} = 1$ and partial ~~filtering~~ penetration of it upwards in the form of a wave of type 1 with reflection at the point $v_{10+} = 1 + u^{\frac{1}{2}}$. As a result, under these conditions, if $u < 1$ and the frequencies are below the critical values, reflected signals reach the earth from all three points $v_{1,2,0\pm}$ ("tripling") instead of the usual two reflected signals (from the points v_{20} and v_{10}). More precisely, there should take place not "tripling" but "multiplication", since a signal reflected from the point v_{10+} will pass through the region $v \cong 1$ only partially, and will be in part reflected from it, and then, following reflection from the point v_{10+}, will again go downward, etc. If the ~~coefficient of~~ transmi~~ssion~~tance for the region $v \cong 1$ is equal to $D = 1 - R = 1 - |\rho|^2$, where $|\rho|$ is the amplitude ~~coefficient of~~ reflect~~ion~~ance, then the intensity of the succe~~eding~~ssive signals reflected from the point v_{10+} to the earth is obviously equal to D^2, $D(D - D^2)$, etc., ~~on~~

In I, Paragraph 3, an evaluation of D in the case $D \ll 1$ was made. Here are encountered considerable calculational difficulties, which will be taken up again at the end of this section. At the same time, calculation of D in the region of small values of α, where

$$|\rho|^2 = R = 1 - D \ll 1, \tag{49}$$

is effected without difficulty, and we shall give it.

For the zero approximation we shall take the solution $F_\pm^{(0)}$ of the equations (11), which is valid for $\alpha = 0$. When

$$\alpha \ll 1, \tag{50}$$

$$F_\pm = E_y \pm jE_z = F_\pm^{(0)} + F_\pm^{(1)}, \quad |F_\pm^{(1)}| \ll |F_\pm^{(0)}|, \tag{51}$$

$F_\pm^{(1)}$, as may be shown from (1) and (2), satisfies ~~satisfying~~ the equations:

$$\frac{d^2F_\pm^{(1)}}{dx^2} + \frac{\omega^2}{c^2}\left(1 - \frac{v}{1 \mp \sqrt{u}}\right)F_\pm^{(1)} = \tag{52}$$

$$= -\frac{\omega^2}{c^2}\alpha^2\left\{\frac{uv(1-u-2v) + \sqrt{u}\, v(uv+u+v-1)}{2(u-1)^2(1-v)}\right\}F_\pm^{(0)},$$

where by virtue of condition (50) $\sin^2\alpha$ is set $= \alpha^2$ and the expansion is performed in (2) with a accuracy ~~precision~~ up to α^2.

Further, to make the problem easier and more concrete, we assume that a non-homogeneous medium occupies the region of values of v between $x = \pm x_0$, so that

$$v = 1 + ax, \quad -x_0 \leqslant x \leqslant x_0, \quad ax_0 \ll 1. \tag{53}$$

When $|x| > x_0$, the medium is homogeneous with indices of refraction:

$$n^2 = n_-^2 \,(v = 1) = u^{\frac{1}{2}}/(1 + u^{\frac{1}{2}}), \qquad (54)$$

and in the zero approximation there is a wave

$$F_-^{(0)} = e^{-j\omega n_- x/c}, \qquad (55)$$

i.e., a wave moving upward of the type of a wave reflected at the point $v_{10+} = 1 + u^{\frac{1}{2}}$ (since $n_-^2 = 0$ ~~for~~ at v_{10+}); due to the linearity of the equations, the amplitude of the wave $F_-^{(0)}$ is not ~~essential~~ important, and is set equal to unity. With all these assumptions, in solving equation (52), we find for $F_-^{(1)}$:

$$F_-^{(1)} = C_1 e^{-j\omega n_- x/c} + C_2 e^{+j\omega n_- x/c} + \frac{K\{\ln x_0 - \ln|x|\}}{2j(\omega/c)n_-} e^{-j\omega n_- x/c} + K e^{j\omega n_- x/c} \int_{-x_0}^{x} \frac{e^{-2j\omega n_- x/c}}{2j(\omega/c)n_- x} dx, \qquad (56)$$

$$K = -\omega^2 u \alpha^2 / 2c^2 (1 + u^{\frac{1}{2}})^2 a.$$

When $x > x_0$ there should be only a penetrating wave, and when $x < -x_0$ an incident and reflected wave, i.e.,

$$x > x_0 : F_- = (1 + A_1) e^{-j\omega n_- x/c};$$
$$x < -x_0 : F_- = e^{-j\omega n_- x/c} + \rho e^{j\omega n_- x/c}. \qquad (57)$$

When $x = \pm x_0$ the field F_- and its first derivative with respect to x must be continuous, whence:

$$C_1 = A_1 = 0, \quad \rho = C_2 = -K \int_{-x_0}^{x_0} \frac{e^{-2j\omega n_- x/c}}{2j(\omega/c) n_- x} dx. \qquad (58)$$

In the approximation being considered the solution does not change in the region $x < x_0$, which is related to the smallness of $|\rho|^2$ and conforms to the law of conservation of energy. The ~~coefficient~~ ~~of~~ reflect~~ion~~ance

$$|\rho| = [\, m\omega^3 u^{3/4}\alpha^2/16\pi e^2 c(1+\sqrt{u})^{3/2}(dN/dx)_{v=1}]\ \mathrm{X}$$

$$\mathrm{X}\left\{\pi + 2si(2\ \ n\_\ x_o/c)\right\},$$

where $si\ t = -\int t^{-1}\sin t\ dt$ is the sine integral and it is ~~taken into~~ considered ~~consideration~~ that

$$a = (4\pi e^2/m\omega^2)(dN/dx)_{v=1},$$

inasmuch as $v = (4\pi e^2/m\omega^2)\ N(x) = (4\pi e^2/m\omega^2)\left\{N(v=1) + (dN/dx)_{v=1}x\right\}$.

Since for large values of the ~~independent variable~~ argument $si\ t \cong -\cos t/t$ and when $t = 0$, $si\ t = si\,0 = -\pi/2$, then $\rho = 0$ when $x_o = 0$, and for large values of $2(\omega/c)n\_\ x_o$ the coefficient $|\rho|$ is a maximum and equal to

$$|\rho_{max}| = \frac{0.82\cdot 10^{-20}\,\omega^3 u^{3/4}\alpha^2}{(1+u^{\frac{1}{2}})^{3/2}(dN/dx)_{v=1}} \qquad (60)$$

When $x_o \sim 10^5$, $2(\omega/c)n\_\ x_o/c \cong 10^2$ (for $\lambda \sim 6\cdot 10^3$) and for $a \sim 10^{-6} - 10^{-7}$, on the one hand, the condition of (53) is fulfilled, and on the other hand: $|\rho_{MAX}| - |\rho| \sim |\rho_{MAX}|/2(\omega/c)n\_x_o \sim |\rho_{MAX}|/100$. Thus, in practice, in the majority of cases $|\rho| \sim |\rho_{MAX}|$. When $u = \frac{1}{4}$, $\omega = 2\omega_H = 1.78\cdot 10^7$ and $|\rho_{MAX}| = 8.5\,\alpha^2/(dN/dx)_{v=1} \sim 10^2\alpha^2$, for $dN/dx \sim 0.1$ (this corresponds to $a \sim 10^{-6}$). Condition (49) denotes that formula (60) is applicable ~~while~~ when $\alpha \ll 10^{-2}$, i.e., in practice while $\alpha < 2$-3 degrees. In order of magnitude $|\rho_{MAX}| \sim 1$ when $\alpha \sim 5$ degrees and thus the "tripling" effect comes into play in the region of ~ 5 degrees from the magnetic poles. However, with the entirely possible, particularly sporadic, value $dN/dx \sim 1$ this region may reach 10-20 degrees, i.e., may be ~~completely~~ quite accessible for observations.

In another boundary case, when

$$D = 1 - |\rho|^2 \ll 1, \qquad (61)$$

the problem may be solved if we select as the zero approximation a standing wave of type 2 which is completely reflected in the region ~~about~~ near the point v_{20}. The corresponding solution may at once be obtained in ~~clear~~ explicit form on the basis of the results of Paragraph 3 [see Paragraph 3, beginning with (32)] and of the solutions for the isotropic case (3,4). In order to obtain the coefficient D it is most simple to use Ritz's method here, as was done in I, but with a certain simplification. Specifically, as the original solution we take:

$$E_{y,z} = E_{y,z,2} + dE_{y,z,1}, \qquad (62)$$

where $E_{y,z,2}$ is a standing wave of type 2, $E_{y,z,1}$ is a wave of type 1 moving upward from the region $v \cong 1$, and d is a physical parameter, (equal evidently to $D^{\frac{1}{2}}$), which must be ~~subject to~~ determin~~ation and evidently $d = D^{\frac{1}{2}}$~~ed. In order that the variation of the ~~operating~~ action integral $I = \int L dx$ ~~on~~ at the upper boundary of the region of integration may ~~change to~~ become zero for any variation of d, it is necessary to choose as the LaGrange function the expression:

$$L = -\tfrac{1}{2}(E'_y E'^*_y + E'_z E'^*_z) + (\omega^2/c^2)(AE_y + iCE_z)E^*_y/2 +$$
$$+ (\omega^2/c^2)(BE_z - iCE_y)E^*_z/2 + \text{k. c.} \qquad (63)$$

The condition of the minimum of $I = \int L dx$ ~~is~~ determines ~~by~~ d in (62). We shall not give the corresponding calculations, since it proves very difficult to carry them to a numerical result; the point is that application of the "pass" method to the integral in the numerator of the expression for d, as was done in I in calculating the pre-exponential ~~multiplier~~ constant, runs into the difficulty that at the point of the pass not only $n_2 = n_1$, but also $[d(n_2 - n_1)/dx] = \infty$. Generally speaking, this fact should not affect the exponential multiplier. Formulas (46) and (47) were obtained in I with a ~~precision~~ accuracy up to ~~the~~ that of the pre-exponential ~~multiplier~~ constant:

$$D \sim e^{-\gamma}; \quad \gamma = 1.06 \cdot 10^{-20}\, \omega^3 \beta \cdot u^{\frac{1}{2}}[\sin^2 \alpha \;/(dN/dx)_{v=1}],$$

where β is a ~~multiplier~~ constant of the order of unity. For the same case as above when $u = \frac{1}{4}$, $\gamma = 17.5 \sin^2 \alpha/(dN/dx)_{v=1}$.

When $(dN/dx)_{v=1} = 0.1$, according to this formula $D = 0.135$ ($\rho = \sqrt{1 - D} \cong 0.93$) for $\alpha = 6$ degrees. Comparing this result with those given above we see that the pre-exponential ~~multiplier~~ constant in I, (46) - (47) cannot be large. From this it follows that under normal conditions when $dN/dx \sim 0.1$ and even when $dN/dx \sim 0.5$, the "tripling" effect cannot be observed at average latitudes. Therefore, the appearance, for example, of three and four signals from the F-layer observed in Tomsk ($\sin^2 \alpha \cong 0.1$) (6) evidently cannot be explained (7) by reflection of waves from the point $v_{10+} = 1 + u^{\frac{1}{2}}$. Reflection from this point (the effect of "tripling" or "multiplication") could have appeared only with gradients of $(dN/dx)_{v=1} \gtrsim 1$, which of course must not be excluded. However, for the reason given as well as for a number of others (8), it seems to us more probable that the triplets and quadruplets which appear at medium and low latitudes may be explained by the influence of a sporadic F_2-layer (8).

Solution of the problem as to what takes place is possible through experimentation as the result of polarization measurements, since in the case of "multiplication" only the first signal -- the signal reflected from the point $v_{10} = 1 - u^{\frac{1}{2}}$ -- is ~~conventional~~ extraordinary in the sense of polarization; in the case of a sporadic layer, if there is a quadruplet, two signals should be ~~conventional~~ ordinary and two should be ~~unconventional~~ extraordinary.

Experimental study of the "tripling" effect, or more precisely the "multiplication" effect, which is possible at ~~great~~ high latitudes, ~~remains so far~~ has not been ~~undone~~ done for the present.

6. The equations (1) and (2) are valid only in the absence of absorption. Within the framework of elementary theory, ~~accounting for~~ absorption is ~~achieved through the~~ taken into consideration by addi~~tion~~ng to the equation of motion of an electron (I, (6)) ~~of~~ the ~~member~~ term $m\nu_э r$, where $\nu_э$ is the effective number of collisions; as a result the equation for r has the form:

$$m\ddot{r} = -eE - m\nu_э\dot{r} - (e/c)[\dot{r}H^{(0)}]. \tag{64}$$

Selecting the z axis as the direction of $H^{(0)}$ and letting $E \sim e^{j\omega t}$ and $r \sim e^{j\omega t}$ for the current $i = j\omega P_t = -j\omega eNr$, we obtain $(\omega_H = eH^{(0)}/mc)$:

$$i_z = \frac{eNE_z}{m(i\omega + \nu_э)}, \quad i_x \pm j i_y = \frac{e^2N(E_x \pm jE_y)}{m(j\omega + \nu_э \mp j\omega_H)} \tag{65}$$

A clear expression for the tensors of the dielectric constant ε_{ik} and the conductivity σ_{ik} are obtained at once from this, if it is ~~borne in mind~~ remembered that by definition

$$i_k = \sigma_{kl} E_l + i\omega\alpha_{kl} E_l$$
$$\varepsilon_{kl} = \delta_{kl} + 4\pi\alpha_{kl}$$

where δ_{kl} is Kroneker's symbol and the summation with respect to l is performed from 1 to 3 ($E_1 = E_x$, etc.). Consideration of the propagation of waves in the presence of absorption reduces essentially to the case of the absence of absorption, since it may be taken into consideration by simple substitution in all the formulas (see (2)) of $\varepsilon'_{ik} = \varepsilon_{ik} - \frac{i4\pi\sigma_{ik}}{\omega}$ for ε_{ik}.

In order to find the magnitude of $\nu_э$ it is necessary to use the kinetic-equation method, which has already been done for the absence of a field. Here we shall make a brief generalization of the corresponding results for the case of the presence of a field. (This problem has already been solved at our suggestion by G. Avak'yanets, and independently by S. Tselishchev. In view of the fact that their results have not been published and are not known to us in precise detail, the calculations have been done anew, and it seems appropriate to give them here.) Representing the distribution function in the form $f = f_o(v) + (V/v)f_1(v)$, where V is the velocity and $|f_1| \ll |f_o|$, we obtain in the usual way (ν is the number of collisions for the velocity v):

$$\partial f_1/dt - (eE/m)\,\partial f_o/\partial v - (e/mc)[Hf_1] + \nu f_1 = 0, \qquad (66)$$

since $[VH]\nabla_v f = [Hf_1](V/v)$. Letting $\partial f_1/\partial t = j\omega f_1$, we find (the magnetic field $H^{(0)}$ was selected as the z-axis):

$$f_{1z} = \frac{eE_z(\partial f_o/\partial v)}{M(j\omega + \nu)}, \quad f_{1x} \pm f_{1y} = \frac{e(E_x \pm iE_y)(\partial f_o/\partial v)}{M(j\omega + \nu \mp j\omega_H)} \qquad (67)$$

The average current density i is equal to $-eN \int V(f_1 \cdot V/v)\,dV\,d\Omega$, whence (see (9)):

$$i_z = \frac{8e^2 N E_z}{3\pi^{1/2} m}\int_0^\infty \frac{u^4 e^{-u^2}\,du}{j\omega + \nu(u)}$$

$$i_x \pm i_y = \frac{8e^2 N(E_x \pm jE_y)}{3\pi^{1/2} m}\int_0^\infty \frac{u^4 e^{-u^2}\,du}{(j\omega + \nu(u) \mp j\omega_H)}, \qquad (68)$$

where it is ~~taken~~ (assumed) that f_0 is the Maxwell distribution with temperature T and $u = (m/kT)^{1/2}v$.

The expression for i_z coincides with the one obtained in the absence of a field, and therefore for $\omega \gg \nu_э$ the values of $\nu_э$, computed when formulas (65) and (68) for i_z and $i_x = j i_y$ are made identical, coincide with what was given in (9). The same thing occurs if

$$\omega - \omega_H \gg \nu_э \qquad (69)$$

~~and~~ with respect to the component $i_x + j i_z$.

If then, the condition of (69) is not met, the values of in i_z, $i_x - j i_y$ and $i_x + j i_y$ are different, i.e., anisotropy of the effective number of collisions occurs (all values of $\nu_э$ are identical also if $\nu_э \gg \omega$). We shall limit ourselves to calculation of $\nu_э$ for the frequency $\omega = \omega_H$ only.

With collision with neutral molecules $\nu(v) = \pi \cdot a^2 N_m v$ (a is the radius of a molecule) and, as may easily be shown, for $i_x + j i_y$

$$\nu_{э,\,\omega=\omega_H} = (3\pi/8)\,\pi a^2 N_M \bar{V}, \qquad (70)$$

where $\bar{v} = (8kT/\pi m)^{1/2}$ -- the ~~average~~ arithmetic mean velocity -- and N_m is the number of molecules per cubic centimeter.

With the condition $\omega \gg \nu_{\ni}$, the expression $\nu_{\ni,\omega \gg \nu_{\ni}} = (4\pi/3)a^2 N_m \bar{v}$, for the absence of a field was obtained in (9); this is also valid for ~~j_z~~ i_z and ~~$j_x - ij_y$~~ $i_x - ji_y$ in (68); the ratio is

$$\nu_{\ni,\omega \gg \nu_{\ni}} / \nu_{\ni,\omega=\omega_H} = 32/9\pi$$

~~During~~ In collision with ions

$$\nu(v) = 2\pi (e^4/m^2v^4) N_i v \ln \left\{ 1 + [N_i^{-2/3}/(e^4/m^3v^4)] \right\}$$

and with the condition that $\ln(2kT/e^2N_i^{1/3}) \gg 1$, for ~~$j_x - ij_y$~~ $i_x - ji_y$:

$$\nu_{\ni,\omega=\omega_H} \cong (\pi^2/16)(e^2/kT)^2 N_i \bar{v} \ln(2kT/e^2N_i^{1/3}), \qquad (71)$$

*

and, thus,

$$\nu_{\ni,\omega \gg \nu} / \nu_{\ni,\omega=\omega_H} = 32/3\pi.$$

We see that with $\omega = \omega_H$ the anisotropy of $\nu_{\ni}$ is rather considerable, and that in (65) it is necessary to use different values of $\nu_{\ni}$ for ~~j_z, $j_x - ij_y$~~ i_z, $i_x - ji_y$ and for ~~$j_x + ij_y$~~ $i_x + ji_y$ ~~with~~ in collisions of ions. When ω is close to ω_H this effect is ~~much~~ less considerable; generally it has no special practical significance, since for $\omega \sim \omega_H$, ~~$j_x + ij_y \gg j_x - ij_y, j_z$~~ $i_x + ji_y \gg i_x - ji_y, i_z$, and, what is most important, it is scarcely possible to determine $\nu_{\ni}$ simultaneously by experimental means for the same region of the ionosphere for different components. As far as the absolute value of $\nu_{\ni}$ is concerned, the presence of the ~~multiplier~~ constant $\cong 3$ ~~does~~ is not ~~have significance~~ important due to lack of sufficiently precise knowledge of the parameters a, N_i, and T in (71) ~~while performing~~ in the experiments.

*while for the isotropic case and for ~~j_z, $j_x - ij_y$~~ i_z, $i_x - ji_y$ in (68), with $\omega \gg \nu_e$

$$\nu_{\ni,\omega \gg \nu_{\ni}} \cong (2\pi/3)(e^2/kT)^2 N_i \bar{v} \ln(2kT/e^2N_i^{1/3})$$

All of the ~~experiments~~ expressions obtained in both the isotropic case and in the absence of a field are based on the assumption that in the ionosphere the ~~active~~ effective field $E_э$ is equal to the average macroscopic field

$$E_э = E. \tag{72}$$

The question as to the validity of this equality has provoked rather widespread discussion and has already been examined by us in detail (10). In the course of this, ~~the conclusion~~ it was ~~reached~~ concluded that we can have complete confidence in equality (72) only upon observance of the condition:

$$e^2N/\omega(\omega - \omega_H) \ll 1. \tag{73}$$

We wish to take the opportunity to note here the possibility of replacing the condition (73) with one incomparably weaker. Condition (73) is obtained in (10) as the result of the requirement of smallness of the first approximation by comparison with the zero approximation, which gives (see (10) for symbols):

$$\xi_{i1} \ll \xi_0 = -eE_0e^{i\omega t}/m\omega^2,$$

where $m\ddot{\xi}_{i1} + e(r_{i1}\nabla)\partial\varphi/\partial x_i = -(e)(\partial^2\varphi_u/\partial x_i^2)\xi_0$. In (10) the conclusion was drawn from this that it is necessary to require observance of the inequality $e(\partial^2\varphi_u/\partial x_i^2)\xi_0 \ll eE_0e^{i\omega t} = m\ddot{\xi}_0$, i.e., the inequality $\ddot{\xi}_{i1} \ll \ddot{\xi}_0$ which also reduces to (73). However, the characteristic frequency of variation of ξ_{i1} is just like that of $\partial\varphi/\partial x_i^2$, i.e., on the order of $\bar{v}/\bar{r}$, where $\bar{v}$ and $\bar{r}$ are the average values of velocity and the ~~mutual~~ distance between the electròn and the ions (the ~~portion~~ part which is not a function of time $\partial^2\varphi_u/\partial x_i^2 = \partial^2\varphi_{u_{MACRO}}/\partial x_i^2 = 0$; see (10)). Therefore $\ddot{\xi}_{i1} \sim (\bar{v}/\bar{r})^2\xi_{i1}$ and from the condition $\ddot{\xi}_{i1} \ll \ddot{\xi}_0$ we derive the inequality

$$\frac{e\partial^2\Phi_0/dx^2}{m(\bar{v}/\bar{r})^2} \sim \frac{e^2 N_i}{m(\bar{v}/\bar{r})^2} \ll 1, \qquad (74)$$

where N_i is the concentration of ions. The presence of a magnetic field does not change anything here. In order of magnitude $\bar{r} \sim N_i^{-1/3}$ and $\bar{v} \sim (kT/m)^{\frac{1}{2}}$, and consequently for $T \sim 300$ degrees Kelvin, $\bar{v} \sim 10^7$, and the condition (74) assumes the form:

$$(e^2/kT)N_i^{1/3} \sim 10^{-5}N_i^{1/3} \ll 1. \qquad (75)$$

Inequality (75) is always fulfilled under the conditions of the ionosphere whence is yielded complete observance of equality (72).

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