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1. Setting up the Problem. In the present article we consider the solution of van der Pol's equation

$$d^2x/dt^2 - n(1 - x^2).dx/dt + x = 0 \qquad (1.1)$$

for large values of the parameter n.

In the xp-phase plane the equation (1.1) is transformed in the form following:

$$pp' - n(1 - x^2)p + x = 0 \qquad (p = dx/dt) \qquad (1.2)$$

where the prime designates differentiation with respect to x.

The solution of this equation possesses the character schematically represented in figure 1 (for the limit cycle). [Figure 1 in the appendix.]

It is well known that in the domain I and in the domain III the solution of equation (1.2) tends correspondingly to the solutions of "abbreviated" equations following:

$$pp' - n(1 - x^2)p = 0 \qquad (1.3)$$

$$- n(1 - x^2)p + x = 0 \qquad (1.4)$$

However, the domains in which these two limit solutions are applicable do not come in contact with each other and therefore it is impossible to join these solutions. It is not known how to select the constant of integration in equation (1.3) in order that during analytical extension of the solution into domain III this solution would pass over into that solution which tends to the solution of the second equation (1.3).

In the present work we introduce two "connecting" domains II and IV for which we establish particular asymptotic solutions of equation (1.2) that differ from the solutions of the "abbreviated" equations (1.3) and (1.4). The domains I, II, III, IV intersect and thus we can find a complete solution for the entire cycle with an accuracy up to magnitudes of any order of smallness relative to n.

2. The Solution for the Domain I. Designating by a_1 and a_2 the values of x for which $p = 0$ (for the limit cycle we have $a_1 = a_2 = a$, where a is the amplitude of stationary self-excited oscillation), we define two parts of the domain I so that we have:

$$-1 + e < x < a_1 - e \qquad e > 0 \text{ and } p > 0$$

$$-a_2 + e < x < +1 - e \qquad p < 0 \text{ and } e > 0 .$$

Obviously it is sufficient to consider the solution in one of the parts of domain I, for example in the first part. We seek the solution in the form:

$$p = n\sum_{m=0}^{\infty} f_m(x).\, n^{-2m} \qquad (2.1)$$

Substituting (2.1) into (1.2) and equating coefficients with identical powers of n, we obtain a recurrence system of equations: (note: "n" not the parameter "nu")

$$f_0' = 1 - x^2, \quad f_0 f_1' = -x, \quad -f_0 f_2' = f_1 f_1', \quad \ldots \quad f_0 f_{n-1}' = -\sum_{k=1}^{m} f_k f_{n-1-k}' \qquad (2.2)$$

the solution of which is elementary. Thus for the first two functions we have:

$$f_0(x) = c + x - x^3/3$$

$$f_1(x) = \frac{x}{x_1^2 - 1}\left[\log(1 - x/x_1) - \tfrac{1}{2}\log\frac{(2x + x_1)^2 + 3(x_1^2 - 4)}{4(x_1^2 - 3)}\right] +$$

$$\frac{x_1^2 - 2}{x_1^2 - 1}\left(\frac{3}{x_1^2 - 4}\right)^{\frac{1}{2}} \cdot \left[\arctan\frac{2x + x_1}{(3x_1^2 - 12)^{\frac{1}{2}}} - \arctan\frac{x_1}{(3x_1^2 - 12)^{\frac{1}{2}}}\right] \qquad (2.4)$$

Here in (2.4) x_1 designates the real (positive) root of the equation

$$f_0(x) = c + x - x^3/3 = 0 \text{ and it is assumed that } c \geq 2/3$$

(which holds for the limit cycle, for example).

The functions $f_n(x)$ possess a singularity in the neighborhood of the point $x = x_1$. From the system (2.2) it is easy to clarify the nature of these singularities, namely: (note: "n" here is not the parameter "nu")

$$f_n(x) \sim [\log(x - x_1)]^{n-1} / (x - x_1)^{n-1} \qquad (2.5)$$

(note: the n's here are meant to be m's, not the parameter "nu".)

Hence it follows that the series (2.1) preserves its asymptotic character up to those values of x satisfying the condition $O(x_1 - x) \geq O(\log n/ n^2)$ [note: the n's here are the parameter "nu"].

In particular, the series (2.1) is an asymptotic series for $x = x_1 - O(1/n)$. (note: "n" here is the parameter "nu".) In this case p will be of the order of unity. This information will be used later on.

Expanding the first three functions $f_n(x)$ in the neighborhood of x_1 we get:

$$f_0(x) = -(x_1^2 - 1)(x - x_1) - x_1(x - x_1)^2 - (x - x_1)^3/3$$

$$f_1(x) = \frac{x_1}{x_1^2 - 1}\log(1 - x/x_1) + g - (x_1^2 - 1)^{-2}(x - x_1) -$$

$$\frac{x_1(x_1^2 - 4)}{6(x_1^2 - 1)^3}(x - x_1)^2 + \frac{x_1^4 - 3x_1^2 - 1}{9(x_1^2 - 1)^4}(x - x_1)^3 \ldots \qquad (2.6)$$

$$f_2(x) = -\frac{x_1^2}{(x_1^2-1)^3}\log\left(1-\frac{x}{x_1}\right)\left[\frac{1}{x-x_1} + \frac{1+g(1+x_1^2)}{x_1^2(x_1^2-1)} + \ldots\right]$$

$$-\frac{(1+g)x_1^2}{(x_1^2-1)^3}\cdot\frac{1}{x-x_1} - \frac{x_1(x_1^2+1)}{2(x_1^2-1)^4}\log^2\left(1-\frac{x}{x_1}\right) + \ldots$$

$$\text{where } g = \frac{x_1}{x_1^2-1}\left[\frac{\sqrt{3}(x_1^2-2)}{x_1\sqrt{x_1^2-4}}\arctan\frac{x_1\sqrt{x_1^2-4}}{\sqrt{3}(x_1^2-2)} - \frac{1}{2}\log\frac{3(x_1^2-1)}{x_1^2-3}\right] \qquad (2.7)$$

We will conduct here a proof of the convergence of the series (2.1). This proof is obtained from a consideration of the solution of (1.2) by the method of successive approximation:

$$p_0' = n(1 - x^2), \; \ldots\ldots, \; p_{m+1} = n(1 - x^2) - x/p_m$$

which converges in the domain I. After which, evaluating the difference $p_{m-1} - p_m$ - we are convinced that the difference possesses the order $1/n^{(2m+1)}$; hence it must be convergent - namely, the expansion (2.1) (at least the asymptotic difference).

3. The Solution for the Domain II. The domain II is the neighborhood of the points $p = 0$, $x = a_1$, $x = -a_2$. Let us consider for the sake of definiteness that part of domain II corresponding to the neighborhood of point $p = 0$, $x = a_1$. Let us introduce the variable $q = -np$ and find x as a function of q. The equation (1.2) is rewritten in the following form:

$$dx/dq = \frac{1}{n^2} \cdot \frac{q}{q(x^2 - 1) - x} \qquad (3.1)$$

Let us represent the solution of this equation in the following form:

$$x = \sum_{m=0}^{\infty} X_m(q) \cdot n^{-2m} \qquad \text{(note: capital x, X, is meant to be the Greek letter "chi".)} \qquad (3.2)$$

Substitution of this expression into the equation (3.1) gives the recurrence system of the equations for the determination of the chi functions $X_m(q)$. We get:

$$X_0' \equiv 0, \qquad X_0 \equiv a_1$$

$$[q(a_1^2 - 1) - a_1]X_1' = q, \qquad [q(a_1^2 - 1) - a_1]X_2' = (1 - 2a_1 q) \cdot X_1 X_1' \qquad (3.3)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$[q(a_1^2 - 1) - a_1]X_{m+1}' = \sum_{k=1}^{m} X_k' \cdot X_{m+1-k} - q \sum_{\alpha,\beta,\gamma} X_\alpha' X_\beta X_\gamma \qquad \left\{ \begin{array}{l} \alpha+\beta+\gamma = m+1 \\ 1 \leq \alpha \leq m \\ 0 \leq \beta \leq m \\ 0 \leq \gamma \leq m \end{array} \right\}$$

The solution of this system is elementary. For the first two functions $X_m(q)$ we get:

$$X_1(q) = \frac{1}{a_1^2 - 1}\left[q + \frac{a_1}{a_1^2 - 1}\log\left(1 - \frac{a_1^2 - 1}{a_1}q\right)\right]$$

$$X_2(q) = \frac{a_1}{(a_1^2 - 1)^4}\left\{(a_1^2 - 1)q\left(q + \frac{a_1^2 + 1}{a_1(a_1^2 - 1)}\right) + \left[\frac{a_1^2 + 1}{a_1^2 - 1} + 2a_1 q - 2(a_1^2 - 1)q^2\right] \cdot \frac{\log[1 - q(a_1^2 - 1)/a_1]}{1 - q(a_1^2 - 1)/a_1} + \frac{3a_1^2 + 1}{2(a_1^2 - 1)} \cdot \log^2\left(1 - \frac{a_1^2 - 1}{a_1}q\right)\right\} \qquad (3.4)$$

The functions $X_m(q)$ possess singularities for $q \to a_1/(a_1^2 - 1)$ and for $q \to \infty$. Let us clarify the nature of these singularities. From the formulas (3.4) it is seen that for $q_1 \to a_1/(a_1^2 - 1)$ the function X_1 possesses a singularity of the form $\log(1 - u)$ and X_2 possesses a singularity of the form $(1 - u)^{-1} \cdot \log(1 - u)$, where $u = q(a_1^2 - 1)/a_1$. Hence from the system (3.3) it's easy to obtain that in general

$$X_m \sim \left[\frac{\log(1-u)}{1-u}\right]^{m-1} \qquad (3.4)$$

Hence it follows that the series (3.2) preserves its asymptotic character up to values of q that satisfy the condition

$$O\left(a_1/(a_1^2 - 1) - q\right) \gg O(\log n / n^2) \qquad \text{(note: "n" is the parameter)}$$

Similarly for large negative values of q we obtain $X_1 \sim q$, $X_2 \sim q^2$ and in general $X_m \sim q^m$. Thus for negative values of q the series (3.2) preserves its asymptotic character up to values of q limited by the inequality $O(q) \angle O(n^2)$. In particular, the asymptotic convergence holds for $q = -n$ $(p = 1)$.

The proof of the convergence of the series (3.1) will not be carried out here. This proof is easily to obtain from the solution of the equation (3.1) by the method of successive approximations, setting:

$$x_0 \equiv a_1 \quad , \quad dx_{m-1}/dq = \frac{1}{n^2} \cdot \frac{q}{q(x_m^2 - 1) - x_m} \quad .$$

In order to join the solutions obtained for the domains I and II it is necessary to determine the constant a_1 relative to a given value of the constant c in (2.3) or, what is the same, relative to the value of x_1. Since the series (2.1) converges asymptotically ## up to values of x for which $p \doteq O(1)$, and for same values the series (2.3) converges, then we can join the solutions (2.1) and (3.2). Setting in (2.1) and (3.2) $p = 1$ $(q = -n)$, we obtain two equations with two unknowns x^* and a_1 :

$$1 = n \sum_{m=0}^{\infty} f_m(x^*)\ n^{-2m} \ , \qquad x^* = \sum_{m=0}^{\infty} X_m(-n) \ . \ n^{-2m} \tag{3.5}$$

The solution is handled thus: from the first equation of (3.5) we find x^* and then from the second equation of (3.5) we find a_1 relative to the found x^* --- which a_1 enters into the expressions for the functions $X_m(q)$.

Substituting for the functions $f_m(x^*)$ their expressions (2.6) we will find x^* by the method of iterations. We desire to obtain x^* with definite accuracy and therefore we shall stop the iteration process at the point where the next iteration will not change the magnitudes of a given order of smallness relative to the parameter n. Thus, for example, the functions written in (2.6) are sufficient for calculation of x^* with an accuracy up to the magnitude of the order of $\log^2 n / n^4$. With an accuracy up to the magnitude of the order of $1/n^3$, we ##### obtain after three iterations the following:

$$x^* = x_1 - \frac{1}{n}\frac{1}{x_1^2 - 1} - \frac{\log n}{n^2} \cdot \frac{x_1}{(x_1^2-1)^2} - \frac{1}{n^2}\left[\frac{x_1}{(x_1^2-1)^2} \log x_1 (x_1^2 - 1) - \frac{g}{(x_1^2-1)} + \frac{x_1}{(x_1^2-1)^3}\right] - \frac{\log n}{n^3} \cdot \frac{2x_1}{(x_1^2-1)^4} + O\left(\frac{1}{n^3}\right). \tag{3.6}$$

Proceeding in exactly the same manner with the second equation of (3.5) we obtain for a_1 the following expression:

$$a_1 = x_1 - \frac{\log n}{n^2} \cdot \frac{2x_1}{(x_1^2-1)^2} - \frac{1}{n^2}\left[\frac{2x_1}{(x_1^2-1)^2} \log (x_1^2 - 1) - \frac{g}{x_1^2 - 1}\right] + O\left(\frac{1}{n^3}\right) \tag{3.7}$$

4. <u>The Solution for Domain III</u>. The domain III is defined by the interval of variation of the variables

$$a_1 - e \searrow x \searrow 1 + e \ , \qquad p \angle 0 \quad \text{and} \quad e \searrow 0$$

$$- a_2 + e \angle x \angle -1 - e \ , \qquad p \searrow 0 \quad \text{and} \quad e \searrow 0 \ .$$

The domain III possesses significance essential for relaxation oscillations in the sense that when the oscillatory system falls into this domain the system passes over with a great degree accuracy into stationary self-excited oscillations. We shall remain at length here to obtain the solution in this domain.

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First of all we shall find the partial solution satisfying the condition $p \to 0$ for $x \to \infty$. Let us designate this solution by $P(x)$. This is exactly that solution for which the solution of the second abbreviated van der Pâhe equatiân represents the main term of the expansion. Let us assume therefore

$$P(x) = -\frac{x}{n(x^2-1)} + \pi(x) \tag{4.1}$$

(we shall consider that part of the domain III for which $p < 0$). Then for the functiân $\pi(x)$ we obtain the following equation:

$$\pi'(x) - \left[n^2\frac{(x^2-1)^2}{x} + \frac{x^2+1}{x(x^2-1)}\right]\pi(x) = -\frac{x^2+1}{n(x^2-1)^2} + \frac{n(x^2-1)}{x}\pi\pi' \tag{4.2}$$

Considering the first part of equation (4.2) as the free term we reduce equation (4.2) to an integral:

$$\pi(x) = F(x)/n^3 - (n/x).(x^2-1).e^{+n^2k(x)}\int_{\infty}^{x} e^{-n^2k(\xi)}.\pi\pi' d\xi . \tag{4.3}$$

where

$$F(x) = \frac{n^2}{x}(x^2-1)e^{n^2k(x)}\int_{x}^{\infty} e^{-n^2k(\xi)}\frac{\xi(\xi^2+1)}{(\xi^2-1)^3}d\xi$$

$$k(x) = \frac{1}{4}x^4 - x^2 + \log x + \frac{3}{4} \tag{4.4}$$

hence $k(1) = 0$, $k(x) \geq 0$ for $x \geq 1$ and $k'(x) = (x^2-1)^2/x \geq 0$.

Furthermore it is easy to see that $F(x) = O(1)$. Actually, by integrating (4.4) once with respect to parts for which we multiply the integrand function and divide by $k'(\xi)$, we obtain

$$F(x) = \frac{(x^2+1)x}{(x^2-1)^4} - \frac{x^2-1}{x}e^{n^2k(x)}\int_{x}^{\infty} e^{-n^2k(\xi)}\frac{6\xi^5+12\xi^3+2\xi}{(\xi^2-1)^6}d\xi = O(1)$$

and moreover since $F(x) \geq 0$ and the integral term in the last expression is positive, we then have

$$F(x) \leq (x^2+1)x/(x^2-1)^4 \tag{4.5}$$

Conducting further integration by parts in equation (4.3) we reduce it to a nonlinear integral equation:

$$\pi(x) = F(x)/n^3 + \frac{n}{2}\,\frac{x^2-1}{x}\,\pi^2(x) + \frac{n^3}{2}\,\frac{x^2-1}{x}\,e^{n^2k(x)}\int_{\infty}^{x} e^{-n^2k(\xi)}(\xi^2-1)^2\pi^2(\xi)\frac{d\xi}{\xi} . \tag{4.6}$$

Finally replacing the sought-for function according to the formula:

$$\omega(x) = 2x.w(x)/(x^2-1) \tag{4.7}$$

(note: the Latin letter "w" is really the Greek letter omega.)

we shall for $w(x)$ the integral equation

$$w(x) = f(x)/n^3 + nw^2(x) + (n^3/x^2).(x^2-1)^2.e^{n^2k(x)} \quad e^{-n^2k(\ )} \quad w^2(\ )d \tag{4.8}$$

where

$$f(x) = (x^2-1)F(x)/2x \tag{4.9}$$

This equation we solve by the method of successive approximations, setting

$$w_1(x) = f(x)/n^3$$

$$\cdots\cdots\cdots\cdots$$

$$w_{n+1}(x) = \frac{1}{n^3}f(x) + nw_n^2(x) + n^3\frac{(x^2-1)^2}{x^2}e^{n^2k(x)}\int_{\infty}^{x} e^{-n^2k(\xi)}\xi\,\omega_m^2(\xi)d\xi . \tag{4.10}$$

(n : parameter "nu")

Designating $\max|f(x)|$ by M and $\max|w_m|$ by Ω_m in the interval $1 + e \leq x \leq \infty$ we will have the evaluation:

$$\Omega_1 = \frac{M}{n^3}$$

$$\cdots\cdots\cdots$$

$$\Omega_{m+1} \leq \frac{M}{n^3} + n\Omega_m^2 + n^3\Omega_m^2 \cdot \max\left| e^{n^2k(x)} \frac{(x^2-1)^2}{x^2} \int_x^\infty e^{-n^2k(\xi)} \cdot \xi \, d\xi \right| .$$

Since $\xi^2/(\xi^2 - 1)^2$ is a monotonically decreasing function, then we have:

$$e^{n^2k(x)} \frac{(x^2-1)^2}{x^2} \int_x^\infty e^{-n^2k(\xi)} \cdot \xi \, d\xi = e^{n^2k(x)} \frac{(x^2-1)^2}{x^2} \int_x^\infty e^{-n^2k(\xi)} k'(\xi) \frac{\xi^2 d\xi}{(\xi^2-1)^2} < \frac{1}{n^2} .$$

Donsequently,

$$\Omega_1 = \frac{M}{n^3}, \ \ldots\ldots, \ \Omega_{m+1} \leq \frac{M}{n^3} + 2n\Omega_m^2 . \qquad (4.11)$$

Let us consider the following series of relations:

$$Y_1 = \frac{M}{n^3}, \ \ldots\ldots, \ Y_{m+1} = \frac{M}{n^3} + 2nY_m^2 .$$

This series is formed when one solves by the method of iterations the following equation:

$$Y = M/n^3 + 2n.Y^2$$

and converges, if this equation possesses # real roots, to the least root. From the condition that the roots be real we obtain the fact that the iteration process converges if $8M/n^2$ is less than 1.

Since we have $f(x) < \frac{(x^2 + 1)x}{2x(x^2 - 1)^3} < \frac{1}{8(x - 1)^3}$,

the condition $8M/n^2 < 1$ can be represented in the form $x - 1 > n^{-2/3}$.

For fulfilment of the condition $8M/n^2 < 1$ we shall have

$$Y_m < Y = 1/4n - \sqrt{(1/16n^2 - M/2n^4)} < 2M/n^3$$

and consequently a fortiori we have $\Omega_m < 2M/n^3$. It immediately follows from the finiteness (## boundedness) of Ω_m that the method of successive approximations is convergent. Actually, from (4.13) we have

$$\max|w_{m+1} - w_m| < 4nY. \max|w_m - w_{m-1}|$$

and consequently the series $w_1 + (w_2 - w_1) + (w_2 - w_3) + \cdots\cdot$

converges if $4nY < 1$; that is, also for $8M/n^2 < 1$ and thus the successive approximations converge uniformally toward the solution of the equation (4.8) when the condition $x - 1 > n^{-2/3}$ is fulfilled.

Finally noting that $w_{m+1} - w_m = 0(1/n^{(2m+3)})$ and that f(x) and each $w_m(x)$ are expanded in asymptotic series in n, we conclude that the functions w(x) and hence $\pi(x)$ are expanded in asymptotic series in powers of 1/n.

This asymptotic expansion is mostly easily obtained immediately from the differential equation (1.2) by setting

$$P(x) \approx -\frac{1}{n}\cdot\sum_{m=0}^{\infty} P_m(x)\cdot n^{-2m} \tag{4.12}$$

which gives

$$P_o(x) = x/(x^2-1), \quad P_1(x) = P_o P_o'/(x^2-1) = -x(x^2+1)/(x^2-1)^4$$

$$\cdots\cdots\cdots \tag{4.13}$$

$$P_{m+1}(x) = (x^2-1)^{-1}\cdot\sum_{k=0}^{m} P_k'(x)\, P_{m-k}(x) .$$

It is easy to convince oneself that the functions $P_m(x)$ possess for $x \to 1$ a singularity of the form $1/(x-1)^{3m+1}$ and consequently the series (4.12) preserves its asymptotic character for the condition $0(x-1) \searrow 0(1/n^{2/3})$. We note that in the passage to the limit of convergence the function $P(X)$ will have the order $1/n^{1/3}$.

Let us seek the solution according to the given initial conditions.

In the previous sections we were able to obtain the solution up to values of p not reaching up to the line $p = -x/n(x^2-1)$ by a magnitude of the order of $\log n/n^3$. In particular we were able to obtain the solution for

$$p = p_o = -\frac{a_1}{n(a_1^2-1)} + n^{-2}$$

Here x will differ from a_1 by a magnitude of the order $\log n/n$.

Thus we must construct in the domain III a solution satisfying the condition for $x = x_o$, $p = p_o$; hence

$$p_o - x_o/n(x_o^2-1) = 0(1/n^2) \searrow 0 .$$

We shall seek the solution in the following form

$$p(x) = P(x) + s(x) \tag{4.14}$$

Then for $s(x)$ (note: "s" stands for the Greek letter sigma) we obtain the following equation

$$P(x)s'(x) + [P(x) + n(x^2-1)]s(x) = -s(x)s'(x) \tag{4.15}$$

which we reduce just as we did for $\pi(x)$ to the integral

$$s(x) = \frac{c}{n^2}\frac{\underline{n}(x_o)}{\underline{n}(x)}e^{-n^2\underline{m}(x)} + \frac{n}{2\underline{n}(x)}\left[s^2(x) - s_o^2 e^{-n^2\underline{m}(x)}\right] + \frac{n^3}{2\underline{n}(x)}e^{-n^2\underline{m}(x)}\int_{x_o}^{x} e^{n^2\underline{m}(\xi)}\frac{\xi^2-1}{\underline{n}(\xi)}s^2(\xi)\,d\xi \tag{4.16}$$

where we use the following designations (note: $\underline{n}$ and $\underline{m}$ are used to distinguish them from the parameter n and m when used in summations as previously.).

$$\underline{m}(x) = \int_{x_o}^{x}\frac{x^2-1}{n\,P(x)}dx, \quad \underline{n}(x) = -n\,P(x), \quad s_o = s(x_o) = \frac{c}{n^2}. \tag{4.17}$$

From the expansion it follows that

$$\underline{m}(x) = O(1) > 0, \quad \underline{n}(x) = O(1) > 0 \tag{4.18}$$

Exactly in the same way one can calculate from the initial conditions for p that $c = 0(1)$ (or even less). Let us assume now that

$$s(x) = 2\underline{n}(x)\cdot e^{-n^2\cdot\underline{m}(x)}\cdot\underline{s}(x) \tag{4.19}$$

(note: the underlined "s" is used to distinguish it from "sigma".)

for $\underline{s}(x)$ we obtain the equation

$$\underline{s}(x) = \phi(x)/n^2 + n\underline{s}^2(x)\cdot e^{-n^2\underline{m}(x)} + n^3\cdot\underline{n}^{-2}(x)\cdot\int_{x_o}^{x} e^{-n^2\underline{m}(\xi)}\,\underline{n}(\xi)(\xi^2-1)\,\underline{s}^2(\xi)\,d\xi \tag{4.20}$$

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(c)

$$Q_0\frac{dQ_2}{du} - \frac{Q_2}{Q_0} = u^2Q_1 - Q_1\frac{dQ_1}{du} \qquad (5.4)$$

$$Q_0\frac{dQ_{m+1}}{du} - \frac{Q_{m+1}}{Q_0} = u^2Q_m - \sum_{k=1}^{m} Q_k \frac{dQ_{m+1-k}}{du}$$

As for the initial conditions for $Q_m(u)$ they are determined later on from the condition ~~####~~ that the solution join the sàlution in the domain III.

Let us find the solution of equation (5.4a). Setting $Q_0 = du/dT$ (note: here the capital letter "T" stands for the Greek letter "tau") and after substitution and integration in succession we obtain:

$$d^2u/dT^2 - 2u\,du/dT + 1 = 0\,, \quad du/dT - u^2 + T = 0 \qquad (5.5)$$

(the constant of integration we shall take equal to zero in view of the arbitrariness in the selection of the variable T). The equation (5.5) is a Riccati equation. It reduces to the equation $d^2v/dT^2 - Tv = 0 \quad (u = -dv/vdT)$

the general solution of which is $v = \sqrt{T}\cdot[c_1K_{1/3}(2/3T^{3/2}) + c_2I_{1/3}(2/3T^{3/2})] \qquad (5.7)$

For conjunction with the solution of P(x) we must require that $Q_0 = du/dT \to 0$ for $u \to \infty$; that is, from (5.5) we obtain that $u \to \infty$ for $T \to \infty$. This condition is satisfied only by the solution $v = c_1\sqrt{T}\cdot K_{1/3}(2/3T^{3/2}) \qquad (5.8)$

Making use of the well-known relations of the Bessel functions we get

$$K'_m(x) = K_{m-1}(x) + K_m(x)m/x\,, \quad K_{-m}(x) = K_m(x)$$

and obtain for u the solution $u = \sqrt{T}\cdot K_{2/3}(2/3T^{3/2})/\,K_{1/3}(2/3T^{3/2}) \qquad (5.9)$

For negative values of T it is more convenient to represent the formula (5.9) in the following form ($T_1 = -T$):

$$u = \sqrt{T_1}\left\{J_{-2/3}(\tfrac{2}{3}T_1^{3/2}) - J_{2/3}(\tfrac{2}{3}T_1^{3/2})\right\}\Big/\left\{J_{1/3}(\tfrac{2}{3}T_1^{3/2}) + J_{-\frac{1}{3}}(\tfrac{2}{3}T_1^{3/2})\right\} \qquad (5.10)$$

According to (5.5) the quantity $Q_0 = u^2 - T$ results immediately; and using the asymptotic expansions for the Bessel functions, it's easy to obtain the asymptotic expansion for $Q(u)$. We have: $u \approx \sqrt{T}\cdot(1 - T^{-3/2} - \ldots)$

$$Q_0 = u^2 - T \approx 1/2\sqrt{T} + \ldots \approx 1/2u$$

A more complete asymptotic expansion is more easily obtained immediately from (5.4a). We have

$$Q_0(u) \approx \frac{1}{2u} - \frac{1}{8u^4} + \frac{5}{32u^7} - \frac{11}{32u^{10}} + \frac{539}{512u^{13}} - \ldots \qquad (5.12)$$

For negative values of T the denominator in the expression (5.10) will be converted to zero. Let us designate by a (note: used for alpha) the least root of the equation:

$$J_{1/3}(2/3T_1^{3/2}) + J_{-1/3}(2/3T_1^{3/2}) = 0.$$

Then $u \to -\infty$ for $T_1 \to a$ $(T_1 < a)$. Further according to (5.6) $u = -dv/vdT$ or $u = dv/vdT_1$; therefore u has for $T_1 = a$ a simple pole with "deduction" equal to unity and consequently $u = 1/(T_1 - a) +$ holomorphic function .

Hence $T_1 = a + 1/u + \ldots$ and consequently for $u \to -\infty$ we have

~~########~~ $Q_0 = u^2 + a + 1/u + \ldots$.

A more detailed calculation gives

$$Q_0(u) = u^2 + a + \frac{1}{u} - \frac{a}{3} - \frac{1}{u^3} - \frac{1}{4u^4} + \frac{a^2}{5u^5} + \frac{7a}{18u^6} - \frac{1}{7}\left(a^2 - \frac{5}{4}\right)\frac{1}{u^7} + \cdots \qquad (5.13)$$

Let us now take up the determmnation of $Q_1(u)$. The general solution ##### of (5.4b) will be :

$$Q_1(u) = \frac{1}{A(u)}\left[c + \int_0^u A(u)(u^2 - u/Q_0)du\right], \quad A(u) = \exp\left(\int_0^u -Q_0^{-2}du\right) \qquad (5.14)$$

For conjunction ## with the solution of P(x) we must require that the quantity $n^{-2/3}Q_1(u)$ be bounded for $u = O(n^e)$.

Using the asymptotic expansion (5.12) it is easy to obtain for the constant of integration c the value

$$c = -\int_0^\infty A(u)\left[u^2 - u/Q_0\right]du \quad .$$

Hence finally we have

$$Q_1(u) = A^{-1}(u)\int_u^\infty A(u)(u/Q_0 - u^2)du \qquad (5.15)$$

For $Q_1(u)$ we have the expansion:

$$(5.16)$$

for $u \to \infty$ $\quad Q_1(u) \approx 1/4 - 1/64u^6 + O(1/U^9)$

for $u \to -\infty$ $\quad Q_1(u) \approx \frac{1}{3}u^3 + b_0 - \frac{a}{6u^2} - \left(\frac{1}{27} + \frac{b_0}{3}\right)\frac{1}{u^3} + \left(\frac{1}{450} + \frac{2}{5}b_0\right)\frac{1}{u^5} + \cdots + -\frac{2}{3}\log|u|\left(1 - \frac{1}{3u^3} + \frac{2a}{5u^5} - \cdots\right) \qquad (5.17)$

where

$$b_0 = \frac{1}{A(-\infty)} \cdot \int_{-\infty}^{\infty} A(u)\left[\frac{u}{Q_0} - \frac{1}{3}\frac{u^3}{Q_0^2} - \frac{2}{3}\frac{u}{u^2 + a/2} + \frac{1}{3Q_0^2}\log\left(u^2 + \frac{a}{2}\right)\right]du. \qquad (5.18)$$

The solutions for the remaining functions will have the form:

$$Q_{m+1}(u) = A^{-1}(u)\int_u^\infty A(u)\left[\sum_{k=1}^{m} Q_k dQ_{m+1-k}/du - u^2 Q_m\right] \cdot du/Q_0 \qquad (5.19)$$

where the constants of integration are determined from the condition of boundedness of the quantities $n^{-(2/3)M}Q_m(u)$ for $u = O(n^e)$.

The asymptotic expressions for the functions $Q_m(u)$ will be $\qquad (5.20)$

for $u \to +\infty$

$$Q_2(u) \approx -\frac{u}{8} + O\left(\frac{1}{u^5}\right), \quad Q_3(u) \approx \frac{u^2}{16} + O\left(\frac{1}{4}\right), \ldots, Q_m(u) \approx (-1)^{m-1}\frac{u^{m-1}}{2^{m+1}}$$

for $u \to -\infty$

$$Q_2(u) = \frac{2}{9}u + b_0^{(2)} + \frac{a}{3u} + O\left(\frac{\log u}{u^2}\right), \quad Q_3(u) = -\frac{u^2}{27} - \frac{2b_0}{81}u + \cdots \qquad (5.21)$$

Generally the main singularity of the functions $Q_m(u)$ for $u \to -\infty$ beginning with $m = 2$ will be $Q_m \sim u^{m-1}$.

These results show that the series (5.3) preserves its asymptotic character up to values of u that bounded by the condition $O(/u/) < O(n^{2/3})$; that is, for values of x satisfying the condition $O(x - 1) < O(1)$ and thus the domains in which solutions (5.3) and (4.12) are applicable interconnect. As we have seen the same holds true for the solutions (5.3) and (2.1).

6. Conjunction of the Solutions for the Domains I and IV. Let us return to the first part of the domain I. We must join the solution of (2.1) with the solution ## of (5.3), in which for the letter u it is necessary to take the quantity $-n^{2/3}(1 + x)$ and set $p = + n^{-1/3}Q(u)$.

First of all we note that since $p > 0$ for $x = -1$ $(u = 0)$ then the constant c in formula (2.3) must be greater than 2/3 (for $c = 2/3$ we will have $f_0(-1) = 0$). Let us set $c = 2/3 + g$ (note: this underlined "g" stands for gamma). # The order of magnitude of g can be determined at once; since $p(-1) = O(1/n^{1/3})$, then ng will be of the order of $1/n^{1/3}$ and consequently we will have $g = O(1/n^{4/3})$.

Let us now clarify up to what negative values of x the expansion (2.1) is applicable. Let us consider the case where $c = 2/3$. Increasing the constant c leads only to improving the convergence. For $c = 2/3$ we have:

$$f_0(x) = 2/3 + x - x^3/3 = (1/3)(x + 1)(2 - x)$$

$$f_1(x) = -\int_0^x x\,dx/f_0 = -1/(x + 1) - (2/3)\cdot\log(x + 1) + (2/3)\cdot\log(1 - x/2)$$

From the system of equations (2.2) it is easy now to find that in the neighborhood of $x = -1$ the main singularity of the functions $f_m(x)$ has the form $f_m(x) \sim (x+1)^{-(3m-1)}$ and consequently the series (2.1) preserves its asymptotic character up to values of x satisfying the condition $O(x + 1) > O(n^{-1/3})$, (which corresponds to the values $u > O(1)$), and thus the domains in which the expansions (5.3) and (2.1) are applicable intersect.

In particular the asymptotic convergence of the expansions (2.3) and (5.3) is ensured for $x = -1 + 1/n^{1/3}$ $(u = \#\#\#\# - n^{1/3})$ and thus the constant of integration c can be determined by equating for $x = -1 + 1/n^{1/3}$ the values of p obtained from the formulas (5.3) and (2.1).

$$n^{-\frac{1}{3}}\sum_{m=0}^{\infty} n^{-\frac{2}{3}m}Q_m(-n^{-\frac{1}{3}}) = n\sum_{m=0}^{\infty} n^{-2m}f_m(-1+n^{-\frac{1}{3}}) \qquad (6.1)$$

We shall not conduct here fairly large-scale computations but merely elementary ones. Using the expansions ## of the functions $Q_m(u)$ for large negative values of u (formulas (5.13), (5.17), (5.21)) for the left side of the equation (6.1) we obtain the expression:

$$p(-1+n^{-\frac{1}{3}}) = n^{\frac{1}{3}} - \frac{1}{3} + a\,n^{-\frac{1}{3}} - n^{-\frac{2}{3}} + b_0 n^{-1} + \left(\frac{a}{3} - \frac{2}{9}\right)n^{-\frac{4}{3}} - \ldots$$
$$- \frac{2}{9}\log n/n\left(1 + \frac{1}{3n} - \ldots\right). \qquad (6.2)$$

On the other side, the expansions of the functions $f_m(x)$ in the neighborhood of $x = -1$ give for the right side of the equation (6.1) the expression :

$$p(-1+n^{-\frac{1}{3}}) = n^{\frac{1}{3}} - \frac{1}{3} - n^{-\frac{2}{3}} + \left(1 + \frac{2}{3}\log\frac{3}{2}\right)n^{-1} - \frac{2}{9}n^{-\frac{4}{3}} - \ldots \qquad (6.3)$$
$$+ \frac{2}{9}\frac{\log n}{n}\left(1 + \frac{1}{3n} + \ldots\right) + g\left(n + \frac{1}{3} - \ldots\right) - g^2\left(\frac{1}{5}n^{\frac{2}{3}} - \ldots\right) + \ldots$$

(The written terms of the expansions are sufficient for the determination of g with an accuracy ## up to the magnitude of the order of $1/n^{8/3}$.)

Equating the expressions (6.2) and (6.3) we obtain the equation for the determination of g . Thus with an accuracy up to the magnitude of the order of $1/n^{8/3}$ we obtain:

$$g = a/n^{4/3} - 4\log n/9n + \left(b_0 - 1 - \frac{2}{3}\cdot\log\frac{2}{3}\right)\cdot n^{-2} + O(1/n^{8/3}). \qquad (6.4)$$

7. Determination of the Amplitude of the Steady-State Self-Excited Oscillations.

After the determination of the constant $c = 2/3 + g$ it is easy to compute the root x_1 of the equation $f_0(x_1) = 0$. The solution of this cubic equation can be represented in the following form:

$$x_1 = 2 + \frac{a}{3} \cdot g - \frac{2}{27} g^2 + \frac{2}{243} \cdot g^3 - \dots$$

and substitution of the value of g gives:

$$x_1 \approx 2 + \frac{a}{3} n^{-\frac{4}{3}} - \frac{4}{27} \frac{\log n}{n^2} + \frac{1}{3}\left(b_0 - 1 - \frac{2}{3} \log \frac{2}{3}\right) n^{-2} + O\left(n^{-\frac{8}{3}}\right) \quad (7.1)$$

after which the equation (3.9) permits one to find the amplitude of self-excited oscillations. Computations give:

$$a \approx 2 + \frac{a}{3} n^{-\frac{4}{3}} - \frac{16}{27} \frac{\log n}{n^2} + \frac{1}{9}\left(3b_0 - 1 + 2 \log 2 - 8 \log 3\right) n^{-2} + O\left(n^{-\frac{8}{3}}\right) \quad (7.2)$$

8. Determination of the Period of Self-Excited Oscillations. The period of self-excited oscillations is computed according to the following formula:

$$T = 2 \int_{-a}^{a} dx/p(x) \qquad (8.1)$$

(note: this T is not the same as that used previously for "tau".)

Let us limit ourselves merely to indicating the method of computing the period without actual## conduction of calculations.

Let us divide the entire interval of integration into five parts corresponding to the various domains:

1) from $-a$ to $-x_2$ according to the domain II (this part of the integral let us designate by T''_2); here x_2 is the value of x obtained according to formula (3.2) for values of q equal, for example, to $(1-n^{-4/3})a/(a^2-1)$.

2) from $-x_2$ to $-(1 + 1/n^{1/3})$ according to the domain III (this part of the integral is designated by the letter T_3).

3) from $-(1+ n^{-1/3})$ to $-(1 - 1/n^{1/3})$ according to the domain IV (this part of integral is designated by the letter T_4).

4) from $-(1 - n^{-1/3})$ to x^* according to the domain I (this part of the integral is designated by the letter T_1); here x^* is determined according to formula (3.6).

5) from x^* to a according to the domain II (this part of the integral is designated by the letter T'_2).

The total period T then will equal $T = 2(T_1 + T'_2 + T''_2 + T_3 + T_4)$ (8.2)

In each of the intervals we substitute in place of p the corresponding expansions (in the domain II replacing the variable of integration by q and in the domain IV by u). Using the evaluations of the singularities of the functions carried out for each domain we easily determine the necessary number of terms of the expansions for obtaining the period T with given accuracy, after which the computations reduce to a computation of the integrals.

Let us conduct the results of computations with an accuracy up to the quantities of the order $1/n$ inclusively:

$$T_1' = \int_{-1+n^{-\frac{1}{3}}}^{0} \frac{dx}{p} \approx n^{-\frac{2}{3}} + \frac{1}{9}\frac{\log n}{n} - \left(1 - \frac{1}{3}\log\frac{3}{2}\right) n^{-1} + O\left(n^{-\frac{4}{3}}\right) \qquad (8.3)$$

$$T_1'' = \int_{0}^{x^*} \frac{dx}{p} \approx \frac{1}{3}\frac{\log n}{n} + \frac{2}{3}\left(1 + \log 3 + \frac{1}{2}\log 2\right) n^{-1} + O\left(n^{-\frac{4}{3}}\right) \qquad (8.4)$$

$$\therefore\ T_1 = T_1' + T_1'' \approx n^{-\frac{2}{3}} + \frac{4}{9}\frac{\log n}{n} + \left(\log 3 - \frac{1}{3}\right) n^{-1} + O\left(n^{-\frac{4}{3}}\right) \qquad (8.5)$$

$$T_2' \approx \frac{1}{3}\frac{\log n}{n} + \frac{1}{3}\log\frac{3}{2}\, n^{-1} + O\left(n^{-\frac{4}{3}}\right) \qquad (8.6)$$

$$T_2'' \approx \frac{4}{9}\frac{\log n}{n} + O\left(n^{-\frac{5}{3}}\log n\right) \qquad (8.7)$$

$$T_3 \approx \left(\frac{3}{2} - \log 2\right) n - n^{\frac{1}{3}} + \frac{1}{3} + \left(\frac{a}{2} - \frac{1}{4}\right) n^{-\frac{1}{3}} + \frac{7}{10} n^{-\frac{2}{3}} - \frac{43}{18}\frac{\log n}{n} + \left(\frac{1}{2}b_0 + \frac{1}{12} + \frac{11}{6}\log\frac{2}{3}\right) n^{-1} + O\left(n^{-\frac{4}{3}}\right). \qquad (8.8)$$

$$T_4 \approx n^{\frac{1}{3}} - \frac{1}{3} + \left(a + \frac{1}{4}\right) n^{-\frac{1}{3}} - \frac{17}{10} n^{-\frac{2}{3}} - \frac{1}{18}\frac{\log n}{n} + \left(\frac{1}{6} - d\right) n^{-1} + O\left(n^{-\frac{4}{3}}\right) \qquad (8.9)$$

where

$$d = \frac{1}{2}\log a - b_0 - \frac{1}{4} + \int_{-\infty}^{0} \left(\frac{u}{Q_0} - \frac{u}{u^2 + a}\right) du + \int_0^{\infty} \left(Q_0 - \frac{1}{2}\frac{u^2}{u^3 + 1}\right) du \qquad (8.10)$$

Consequently for the total period of oscillation we obtain:

$$T \approx (3 - 2\log 2)n + 3a/n^{1/3} - 22\log n/9n + (3\cdot\log 2 - \log 3 - 1/6 + b_0 - 2d)\cdot n^{-1} + O(1/n^{4/3}). \qquad (8.11)$$

Let us determine the numerical values of the coefficients of main formulas:

alpha $a = 2.338107 \qquad b_0 = 0.1723 \qquad d = 0.4889$

$$p(0) \approx 2n/3 + 2.338107/n^{1/3} - \frac{4}{9}\frac{\log n}{n} - 1.0980/n + O\left(n^{-5/3}\right)$$

Latin $a \approx 2 + 0.779369/n^{4/3} - \frac{16}{27}\frac{\log n}{n^2} - 0.8762/n^2 + O\left(n^{-8/3}\right)$

$$T \approx 1.613706n + 7.01432/n^{\frac{1}{3}} - \frac{22}{9}\frac{\log n}{n} + 0.0087/n + O\left(n^{-\frac{4}{3}}\right).$$

We have carried out here an asymptotic solution of the simplest equation of non-linear oscillations in order that we might carried out to the end the calculations. But the method used here of introducing "connecting" domains is not limited to just this partial case and can be employed for a more general equation of the type:

psi

$$d^2x/dt^2 - n\Psi(x, dx/dt) + \phi(x) = 0$$

for certain limitations on the function $\Psi(x,p)$. The connecting domains will be the neighborhoods of the lines $\Psi(x, p) = 0$. If in the junction point of the line $\Psi(x,p) = 0$ the expansion of the psi function Ψ in a Taylor series begins with the term $a(x - x_0)p$ where a is a numerical coefficient then the main solution for the domain IV remains without variation, since setting in the neighborhood of the juncture point $\pm(x - x_0)m^{2/3} = u, \quad p = \pm m^{-1/3}Q(u)$

we obtain for the main term of the expansion the equation

$$Q_0 Q_0' - auQ_0 + \phi(x_0) = 0$$ (note: "a" here is a Latin letter)

which reduces to (5.4a) by the simple substitution

$$Q_0 = \left[2\phi^2(x_0)/a\right]^{1/3} Q_0^*, \quad u^* = \left[a^2/\,4\phi(x_0)\right]^{1/3} u .$$ ("a" is Latin)

9. An Example. In conclusion we give an example of solutions of the equation for $n = 10$. For this case the asymptotic formulas deduced give

$p(0) = 7.540$, $a = 2.0138$("a" is Latin) , $T = 18.831$. In figure 2 is the graph of the function $p(x)$, in which two terms of the expansion for each domain were taken in the computations:

for domain I: $p(x) \approx nf_0(x) + f_1(x)/n$

for domain II: $x \approx a + X_1(q)/n^2$ (note: "X" is chi)

for domain III $p \approx P_0(x)/n + P_1(x)/n^3$

for domain IV: $p \approx Q_0(u)/n^{1/3} + Q_1(u)/u$

The values of the functions $Q_0(u)$ and $Q_1(u)$ are given in table 1.

In figure 2 the computed points corresponding to the formulas of domains I and III are drawn as as circular poonts and the points corresponding to the formulas of the domains II and IV are represented by crosses.

The scale for p on the interval from $-a$ to -1 is magnified ten times in comparison with the scale of the remaining portion of the curve.

Submitted 17 April 1947

(English-language summary of the Original)

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The entire cycle of the oscillation in the coordinate system p,x (where $p = dx/dt$) is divided into four parts (domains I, II, III, IV in figure 1). An symptotic series in terms of powers of $1/n$ is established for each domain. For domain I the expansion (2.1) holds where the functions $f_m(x)$ are determined from the recurrent system (2.2) . The seies (3.1) holds for domain II , functions $X_m(q)$ (where "X" stands for chi and $q = -np$) being determined from system (3.4) . The seies (4.17) is valid for domain III, dunctions $P_m(x)$ being determined from (4.18) and the seies (5.3) holds for domain IV, where functions $Q_m(u)$ (where $u = n^{2/3}(x - 1)$, $p = -Q/n^{2/3}$) are determined from the system (5.4) .

The intervals of the variable x for which the different asymptotic series are valid cover each other making it possible to connet the solutions obtained for different domains (that is. to determine the constants of integration entering into the expressions for functions $f_m(x)$ ~~#####~~ and $X_m(q)$) .

The asymptotic expressions for the amplitude of steady oscillation (formula (7.2) and for the period of oscillation (formula (8.12) may now be found.

-end-

Footnote:

I: $v = m\left[f_0(x) + f_1(x)m^{-2} + f_2(x)m^{-4} + \cdots\right]$; $f_0(x) = c + x - \frac{x^3}{3}$

$$f_1(x) = \frac{x}{x_1^2-1}\left[\ln\left(1-\frac{x}{x_1}\right) - \frac{1}{2}\ln\frac{(2x+x_1)^2+3(x_1^2-4)}{4(x_1^2-3)}\right] + \frac{x_1^2-2}{x_1^2-1}\cdot\left(\frac{3}{x_1^2-4}\right)^{\frac{1}{2}}\left[\arctan\frac{2x+x_1}{(3x_1^2-12)^{\frac{1}{2}}} - \arctan\frac{x_1}{(3x_1^2-12)^{\frac{1}{2}}}\right]$$

(x_1: root of $f_0(x) = c + x - \frac{x^3}{3} = 0$, $c > \frac{2}{3}$)

II: $x = X_0(-mp) + X_1(-mp)m^{-2} + X_2(-mp)m^{-4} + \cdots$

where $X_0 = a_1$

$$X_1 = \frac{1}{a_1^2-1}\left[-mv + \frac{a_1}{a_1^2-1}\ln\left(1 - \frac{a_1^2-1}{a_1}(-mv)\right)\right]$$

$$X_2 = \frac{a_1}{(a_1^2-1)^4}\left\{(a_1^2-1)(-mv)\left[-mv + \frac{a_1^2+1}{a_1(a_1^2-1)}\right] + \left[\frac{a_1^2+1}{a_1^2-1} + 2a_1(-mv) - 2(a_1^2-1)(-mv)^2\right]\left[\frac{\ln\left(1-(-mv)\frac{(a_1^2-1)}{a_1}\right)}{1-(-mv)\frac{(a_1^2-1)}{a_1}} + \cdots\right] \frac{3a_1^2+1}{2(a_1^2-1)}\ln^2\left(1-\frac{a_1^2-1}{a_1}(-mv)\right)\right\}$$

(where a_1; value for which $v = 0$ limit eq. $(a_1 = 0)$)

III $v = -\frac{1}{m}\left[P_0(x) + P_1(x)\cdot m^{-2} + P_2(x)\cdot m^{-4} + \cdots\right]$

$P_0(x) = \frac{x}{x^2-1}$ $P_2(x) = \cdots$ etc

$P_1(x) = -\frac{x(x^2+1)}{(x^2-1)^4}$

 (continued)

we obtain for the main term of the expansion the equation

$$Q_o Q_o' - auQ_o + \phi(x_o) = 0 \quad \text{(note: "a" here is a Latin letter)}$$

which reduces to (5.4a) by the simple substitution

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The entire cycle of the oscillation in the coordinate system p,x (where $p = dx/dt$) is divided into four parts (domains I, II, III, IV in figure 1). An symptotic series in terms of powers of $1/n$ is established for each domain. For domain I the expansion (2.1) holds where the functions $f_m(x)$ are determined from the recurrent system (2.2) . The seies (3.3) holds for domain II , functions $X_m(q)$ (where "X" stands for chi and $q = -np$) being determined from system (3.4) . The seies (4.17) is valid for domain III, dunctions $P_m(x)$ being determined from (4.18) and the seies (5.3) holds for domain IV, where functions $Q_m(u)$ (where $u = n^{2/3}(x - 1)$, $p = -Q/n^{2/3}$) are determined from the system (5.4) .

The intervals of the variable x for which the different asymptotic series are valid cover each other making it possible to connet the solutions obtained for different domains (that is. to determine the constants of integration entering into the expressions for functions $f_m(x)$ ~~#####~~ and $X_m(q)$) .

The asymptotic expressions for the amplitude of steady osillation (formula (7.2) and for the period of oscillation (formula (8.12) may now be found.

Footnote:

I: $v = m\left[f_o(x) + f_1(x)m^{-2} + f_2(x)m^{-4} + \cdots\right]$; $f_o(x) = c + x - \frac{x^3}{3}$

-end-

$$f_1(x) = \frac{x}{x_1^2 - 1}\left[\ln\left(1 - \frac{x}{x_1}\right) - \frac{1}{2}\ln\frac{(2x + x_1)^2 + 3(x_1^2 - 4)}{4(x_1^2 - 3)}\right] + \frac{x_1^2 - 2}{x_1^2 - 1}\cdot\left(\frac{3}{x_1^2 - 4}\right)^{\frac{1}{2}}\left[\arctan\frac{2x + x_1}{(3x_1^2 - 12)^{\frac{1}{2}}} - \arctan\frac{x_1}{(3x_1^2 - 12)^{\frac{1}{2}}}\right]$$

(x_1: root of $f_o(x) = c + x - \frac{x^3}{3} = 0$, $c > \frac{2}{3}$)

II: $x = X_o(-m\nu) + X_1(-m\nu)m^{-2} + X_2(-m\nu)m^{-4} + \cdots$

where $X_o = a_1$

$$X_1 = \frac{1}{a_1^2 - 1}\left[-m\nu + \frac{a_1}{a_1^2 - 1}\ln\left(1 - \frac{a_1^2 - 1}{a_1}(-m\nu)\right)\right]$$

$$X_2 = \frac{a_1}{(a_1^2 - 1)^4}\left\{(a_1^2 - 1)(-m\nu)\left[-m\nu + \frac{a_1^2 + 1}{a_1(a_1^2 - 1)}\right] + \left[\frac{a_1^2 + 1}{a_1^2 - 1} + 2a_1(-m\nu) - 2(a_1^2 - 1)(-m\nu)^2\right]\cdot\left[\frac{\ln\left(1 - (-m\nu)\frac{(a_1^2 - 1)}{a_1}\right)}{1 - (-m\nu)\frac{(a_1^2 - 1)}{a_1}}\right] + \frac{3a_1^2 + 1}{2(a_1^2 - 1)}\ln^2\left(1 - \frac{a_1^2 - 1}{a_1}(-m\nu)\right)\right\}$$

(where a_1: value for which $v = 0$ limit e.g. $(a_1 = 0)$

III $v = -\frac{1}{m}\left[P_o(x) + P_1(x)\cdot m^{-2} + P_2(x)\cdot m^{-4} + \cdots\right]$

$P_o(x) = \frac{x}{x^2 - 1}$ $\quad P_2(x) = \cdots$ etc

$P_1(x) = -\frac{x(x^2 + 1)}{(x^2 - 1)^4}$

[end of footnote]

IV $\frac{1}{\ }\cdot Q(u)$ $\left[u = m^{2/3}(x - 1)\right]$

Table 1: Table of Functions $Q_0(u)$ and $Q_1(u)$

u	$Q_0(u)$	$Q_1(u)$	u	Q_0	Q_1
−6.0	38.1747	−73.0343	0.0	1.0187	0.1869
−5	27.1436	−42.5848	0.2	0.8424	0.2149
−4	18.0985	−22.1123	0.4	0.7018	0.2310
−3.8	16.5269	−19.0382	0.6	0.5904	0.2398
−3.6	15.0342	−16.2668	0.8	0.5023	0.2445
−3.4	13.6203	−13.7819	1.0	0.4331	0.2471
−3.2	12.2848	−11.5673	1.2	0.3778	0.2485
−3.0	11.0276	−9.6070	1.4	0.3334	0.2492
−2.8	9.8484	−7.8847	1.6	0.2974	0.2495
−2.6	8.7469	−6.3844	1.8	0.2678	0.2496
−2.4	7.7225	−5.0898	2.0	0.2432	0.2497
−2.2	6.7749	−3.9849	2.2	0.2225	0.2498
−2.0	5.9032	−3.0532	2.4	0.2049	0.2499
−1.8	5.1068	−2.2790	2.6	0.1897	0.2499
−1.6	4.3845	−1.6458	2.8	0.1766	2500
−1.4	3.7351	−1.1379	3.0	0.1652	0.2500
−1.2	3.1568	−0.7393	3.2	0.1551	0.2500
−1.0	2.6476	−0.4344	3.4	0.1461	0.2500
−0.8	2.2047	−0.2080	3.6	0.1381	0.2500
−0.6	1.8249	−0.0457	3.8	0.1310	0.2500
−0.4	1.5041	+0.0664	4.0	0.1245	0.2500
−0.2	1.2372	0.1403			
0.0	1.0187	+0.1869			

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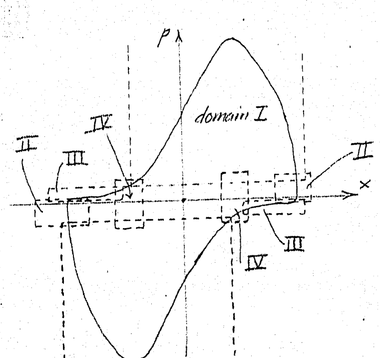

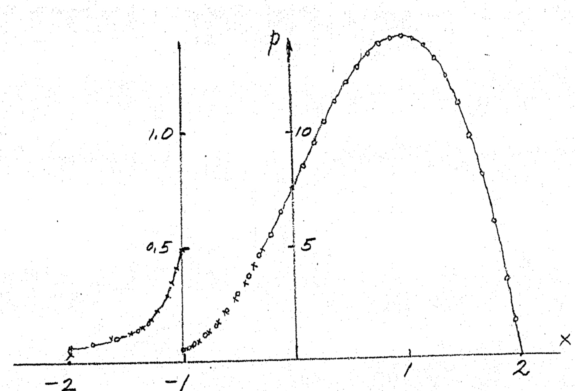

Figures 1 & 2.

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