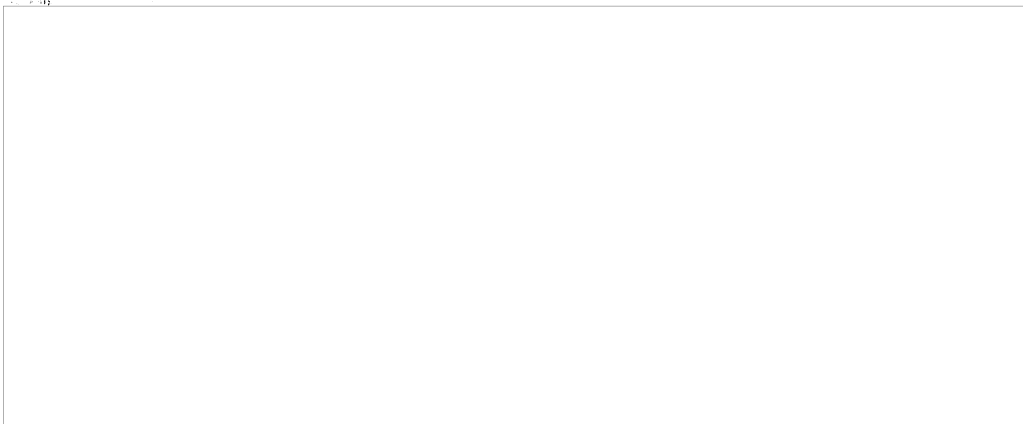


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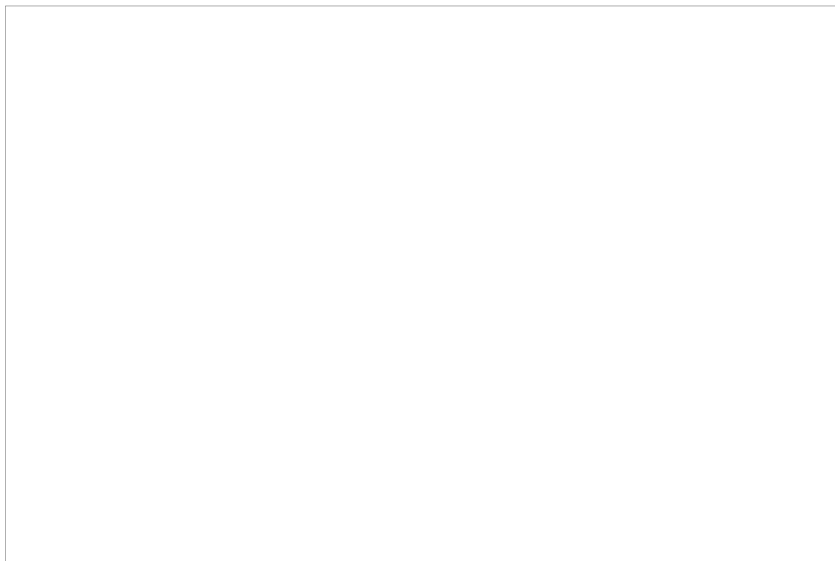
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EQUATIONS OF MOTION OF A VISCOUS GAS

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(Certain calculations associated with the accomplishment of this work were performed by I.A. Smirnova and I.Z. Kalishevich.)

The present work is devoted to the derivation of differential equations for the motion of a viscous gas; for this purpose it is assumed that the gas may, with a sufficient degree of accuracy, be considered a perfect gas -- that is, excluded from consideration are the motions of highly compressed gases -- and that the ratio of the mean free path of the molecules to the characteristic dimension of the phenomenon [being considered] is small in comparison with unity -- that is, the motions of highly rarefied gases are excluded.

It is assumed, moreover, that for the gas motions under consideration the law of equipartition of internal energy among the degrees of freedom of motion of the molecules is fulfilled with a sufficient degree of accuracy; that is, excluded from consideration are cases of motion of a gas with extremely rapid fluctuations in its hydrodynamic parameters in space and time.

The contrast between the regularity of the macroscopic motion of a real gas and the chance nature of its microscopic motions is associated with the fact that the macroscopic motion originates in microscopic motions of a vast number of molecules. From the authenticity of this empirical fact it follows that the finite number of

The chance microscopic motions is so large that the macroscopic motion turns out to be governed by the same statistical laws as would obtain if the number of microscopic motions were infinitely large.

This permits us to discuss, in place of a real, discrete gas with a finite number of finite molecules, a "saturated" gas in the form of an uninterrupted continuum of infinitely small molecules.

Such a model is an approximate scheme, within which the regularity of the macroscopic motion of a gas is retained by way of omitting the possibility of investigating every kind of minute fluctuation associated with the finiteness of the number of molecules in a real gas.

Having replaced the study of a gas with the study of a material continuum, one may introduce the conceptions of the density, velocity, and total and internal energy of a unit mass of gas. Let us take note that, by introducing into consideration a gaseous continuum and by specifying the fundamental hydrodynamic parameters, we shall nowhere create obstacles to the consideration of the microscopic motions of the molecules and shall even, on the contrary, assume their existence. The gaseous continuum so introduced can provide for all the physical characteristics of a real gas which are not related to the finiteness of the number of its molecules. In particular, one may require that the mean free path of a particle of the gaseous continuum be, as in a

real gas, finite, although indeed of extremely small magnitude.

The derivation of equations of motion of a viscous gas presented below reveals that the equations [1, 2, 3, 4] which are customarily applied have been obtained from inadequate physical representations, and in them, therefore, a series of terms of the same order of smallness as those which are retained, is absent.

1. BASIC CONCEPTIONS

Let us designate the mass of gas in a certain volume as m . Then we shall call the density ρ of the gas at a given point M at a given moment of time t the limit of the ratio of mass m at moment t to the volume P , if P , which includes M , is contracting into this point.

Let us designate as K the momentum of the gas contained in volume P . Then we shall call the velocity v of the gas at the given point M at the given moment of time t the limit of the ratio of momentum K at moment t to mass m at moment t , if volume P , which includes M , is contracting into this point.

Since the momentum of a system is equal to the momentum of its center of inertia, in which the total mass of the system is concentrated, the introduced velocity v of the gas is the velocity of the center of inertia of an infinitely small volume.

Let us introduce the conceptions of the total energy U of a unit mass of gas and the internal energy E of a unit mass of

gas.

Let us again consider a certain volume P which includes point M . In volume P there will be a certain mass of gas, possessing a certain finite energy; since by hypothesis the gas is perfect, all the energy of the molecules will be kinetic. Let us call it U_1 . Then we shall agree to call the total energy U of a unit mass of gas at a given point M at a given moment of time t the limit of the ratio of U_1 at moment t to the mass of gas m at moment t , if the volume P , which includes M , is contracting into this point.

On the basis of the Koenig theorem, quantity U_1 may be divided into two components: the kinetic energy U_2 of the center of inertia and the kinetic energy U_3 of the motion relative to the center of inertia.

Let us call the internal energy E of a unit mass of gas at a given point M at a given moment of time t the limit of the ratio of the kinetic energy U_3 (at moment t) of the motion of the molecules relative to the center of inertia, to the mass m of gas (at moment t), if P , which includes M , is contracting into this point.

If we designate as dm , dK , dU^* and dE^* the mass, momentum, total and internal energy of the gas contained in the elementary volume dP , then these formulas follow immediately from the introduced definitions:

$$\begin{aligned}
 dm &= \rho dP \\
 dK &= \rho v dP \\
 dU^* &= \left(\frac{1}{2} \rho v^2 + \rho E \right) dP \\
 dE^* &= \rho E dP
 \end{aligned}
 \tag{1.1}$$

The above introduced magnitude E of the internal energy of a unit mass of gas may obviously be divided into two components.

The first of them, E_1 , will correspond to that part of the kinetic energy of the motion of the molecules which is associated with their forward motion.

The second of them, E_2 , will correspond to that part of the kinetic energy of the motion of the molecules which proceeds from their rotational and vibrational motions.

It is customary in the kinetic theory of gases to designate as the temperature a magnitude proportional to E_1 . The factor of proportionality obviously depends on the unit in which the temperature is measured and becomes completely determined if such a unit has been chosen. Customarily the temperature is accepted as being measured in Kelvin degrees and is designated as T .

For such units of temperature measurement we have by definition

$$m_0 E_1 = \frac{3}{2} k T, \quad k = 1.37 \times 10^{-16} \frac{\text{ergs}}{\text{degrees}} \quad (1.2)$$

where k is the so-called Boltzmann constant, and m_0 is the mass of one molecule of gas.

Since the principle of equipartition of energy among the degrees of freedom applies for the gas motions under consideration, the component E_2 of the quantity E is also found to be proportional to the temperature T .

Consequently we have

$$E = c_v T \quad (1.3)$$

where c_v is a new coefficient of proportionality designated as the specific heat for a constant volume.

2. TRANSFER PHENOMENA AND SOME CONCLUSIONS

We shall give an elementary interpretation of certain physical phenomena belonging to the group of so-called transfer phenomena.

1. Density Self-Diffusion

Let us consider a certain stationary region ABCD of area

ΔS with a normal \mathbf{n} in a macroscopically quiescent gas. Let us designate as λ the mean free path of the molecules and as c the average magnitude of the velocity of thermal motion of the molecules, and let us suppose the temperature of the gas to be constant.

Let us simplify the representation of the motion of the molecules and assume that half of the molecules have a velocity in the direction of the normal \mathbf{n} and half in the contrary direction.

Let us assume, moreover, that all the molecules traverse the path λ without collision in the course of time

$$\Delta t = \frac{\lambda}{c} \quad (2.1)$$

Then half the mass of a layer of gas of thickness λ adjoining area ΔS at the top will move downward, and half the mass of a layer of gas of thickness λ adjoining area ΔS at the bottom will move upward.

If the magnitude of λ is small in comparison to the characteristic dimension l of the phenomenon, we may, with a sufficient degree of accuracy, write

$$\Delta m_2 = \Delta S \lambda \left(\rho + \frac{\partial \rho}{\partial n} \frac{\lambda}{2} \right), \quad \Delta m_1 = \Delta S \lambda \left(\rho - \frac{\partial \rho}{\partial n} \frac{\lambda}{2} \right) \quad (2.2)$$

where Δm_2 is the mass of the layer of gas of thickness λ which is situated above area ΔS , Δm_1 is the mass of the layer of gas of thickness λ which is situated below area ΔS , ρ is the density of gas at a certain point of the area, and $\partial \rho / \partial n$ is the derivative, in the direction of \mathbf{n} , of the density ρ at a certain point of the area ΔS .

It is evident that the magnitude

$$\Delta m = \frac{1}{2}(\Delta m_2 - \Delta m_1) = \frac{1}{2} \Delta S \lambda^2 \frac{\partial \rho}{\partial n} \quad (2.3)$$

gives the mass of gas transferred as a consequence of the variation of density through a region of area ΔS during time Δt .

Let us designate as Q_{np} that flow of mass (through the area with normal \mathbf{n} , in the direction contrary to \mathbf{n}) which is associated with the variation of density.

Then we have

$$Q_{np} = \frac{\Delta m}{\Delta S \Delta t} = \frac{1}{2} \lambda c \frac{\partial \rho}{\partial n} \quad (2.4)$$

Due to the simplified representation of the motions of the gas molecules it is impossible to trust the accuracy of the numerical factor in formula (2.4). It is therefore advisable to put

$$Q_{np} = f_1 \lambda c \frac{\partial \rho}{\partial n} \quad (2.5)$$

where f_1 is a nondimensional factor of the order of unity.

From formula (2.5) we see that with variation of density in a macroscopically quiescent gas there occurs a flow of mass through an area which is stationary relative to the gas. This phenomenon of flow of mass as a result of density variation may for expediency be called density self-diffusion.

It is evident that the reasoning also holds if we consider a macroscopically moving gas and an area moving in space with the macroscopic velocity \mathbf{v} of the gas. The density self-diffusion flow of mass, Q_{np} , will be given by formula (2.5) in this case also.

Let us remark that density self-diffusion has never been taken into account in the writing of equations of motion of a viscous gas.

2. Density Heat Transfer

Let us consider, as above, a certain stationary region of area ΔS with normal \mathbf{n} in a macroscopically quiescent gas, retain the former designations and analyze the problem of transfer of internal energy through area ΔS , assuming the temperature

of the gas to be constant.

A mass of gas $\frac{1}{2} \Delta m_2$ with internal energy ΔE_2 will be transferred downward, and a mass $\frac{1}{2} \Delta m_1$ with internal energy ΔE_1 will be transferred upward. We have, obviously

$$\Delta E_2 = \frac{1}{2} \Delta m_2 c_v T, \quad \Delta E_1 = \frac{1}{2} \Delta m_1 c_v T \quad (2.6)$$

The magnitude

$$\Delta E = \Delta E_2 - \Delta E_1 \quad (2.7)$$

gives the quantity of internal energy transferred as a result of the variation of density through the area ΔS in the course of time Δt .

Let us designate as t_{np} the flow of internal energy (through the area with normal \mathbf{n} , in the direction contrary to \mathbf{n}) associated with the variation of density. We have

$$t_{np} = \frac{\Delta E}{\Delta S \Delta t} = f_1 \lambda c_v T \frac{\partial \rho}{\partial n} \quad (2.8)$$

If one cannot guarantee the accuracy of the numerical factor, it is advisable to put

$$t_{np} = f_2 \lambda c_v T \frac{\partial \rho}{\partial n} \quad (2.9)$$

where f_2 is some numerical factor of the order of unity.

It is evident that the reasoning also applies if we consider a macroscopically moving gas and an area ΔS which is moving with the macroscopic velocity \mathbf{w} of the gas. Here too formula (2.9) will give us the flow of internal energy associated with the variation of density.

We may, for expediency, call the phenomenon of transfer of energy through an area which is moving with the gas, which phenomenon proceeds from the variation of density, density heat transfer.

Let us remark that the phenomenon of density heat transfer has never been taken into account in the derivation of equations of motion of a viscous gas.

3. Viscosity

Using the same simplified schematization of the motion of molecules, it is not difficult to analyze the problem of transfer of momentum through an area which is moving with the gas for cases where the macroscopic velocity of the gas is variable in space.

For the coefficients of viscosity μ_1 and μ which appear here, standing for the derivatives of the components of velocity with respect to the coordinates in expressions for flows of a quantity of energy, we obtain the formulas

$$\mu_1 = f_3 \rho \lambda c, \quad \mu = f_4 \rho \lambda c \quad (2.10)$$

where f_3 and f_4 are numerical factors of the order of unity.

Remark. The discussed phenomena do not exhaust the list of phenomena of transfer, if only for the reason that the usual temperature heat transfer is not among them.

Nevertheless, from the analysis of these three phenomena we may already make three essential conclusions.

First, for the construction of equations of motion of a viscous gas the usual scheme of reasoning with a fluid volume of constant mass cannot be used, since the mass of a volume bounded by a closed surface which is moving with the gas may change as a result of self-diffusional flows of mass.

Second, in the construction of equations of motion of a viscous gas one may not confine oneself to the calculation of viscosity and the usual temperature heat transfer alone, since other transfer phenomena also occur.

Third, in the construction of equations of gas motion one ought to bear in mind that with variability of the hydrodynamic parameters in space, mass, momentum and energy are transferred through a surface which is moving with the gas.

3. GENERAL WRITING OF THE LAWS OF VARIATION

Let us consider, within a space occupied by a medium which is moving with velocity \mathbf{v} , a certain stationary volume V bounded by a surface S , and let us suppose that located among the points of the moving medium which fills the space is a certain

scalar or vector quantity A , which is a function of coordinates and of time.

Together with quantity A let us consider a quantity Φ , which is determined by formula

$$\Phi = \iiint_V A dV \quad (3.1)$$

where dV is an element of volume V .

For the fixed volume V the quantity Φ will be a function only of one time t , and then, obviously, we shall have the formula

$$\frac{d\Phi}{dt} = \iiint_V \frac{\partial A}{\partial t} dV \quad (3.2)$$

Let us grant that the variation of quantity Φ with time arises solely as a result of the independent action of the following two factors:

(1) inside volume V there occurs a growth of quantity Φ with the volume velocity B , leading to the condition that, as a result of the action of this factor in volume dV in the course of time dt , the quantity Φ undergoes a change $\Delta_1 \Phi$, which is determined by the formula

$$\Delta_1 \Phi = B dV dt \quad (3.3)$$

(2) through the surface S of volume V there occurs a flow of quantity Φ with the surface density G_n , leading to the condition that, as a result of the action of this factor upon an element of the surface dS with the external normal \mathbf{n} , in the course of time dt , the quantity Φ undergoes a change $\Delta_2 \Phi$, which is determined by the formula

$$\Delta_2 \Phi = G_n dS dt \quad (3.4)$$

Since the two factors act independently of one another, when we have integrated magnitude $\Delta_1 \Phi$ with respect to volume V and magnitude $\Delta_2 \Phi$ with respect to surface S , have combined the results of the integrations and divided by dt , we obtain a second expression for $d\Phi/dt$, which is given by the formula

$$\frac{d\Phi}{dt} = \iiint_V B dP + \iint_S G_n dS \quad (3.5)$$

Setting (3.2) equal to (3.5), we arrive at the equality

$$\iiint_V \frac{\partial A}{\partial t} dP = \iiint_V B dP + \iint_S G_n dS \quad (3.6)$$

Let us agree to conduct all analyses in arbitrary orthogonal curvilinear coordinates q_1, q_2, q_3 , connected with the Cartesian coordinates by functions which do not include time, and let us select out of volume V a volume bounded by the surfaces

$$q=a, q_1=q_1; \quad q_2=b, q_2=q_2; \quad q_3=c, q_3=q_3 \quad (3.7)$$

If we designate the coefficients of Lamé as H_1, H_2 and H_3 , then for the selected volume the equality (3.6) will be rewritten in the following manner:

$$\begin{aligned} \int_a^{q_1} \int_b^{q_2} \int_c^{q_3} \frac{\partial A}{\partial t} H_1 H_2 H_3 dq_1 dq_2 dq_3 &= \int_a^{q_1} \int_b^{q_2} \int_c^{q_3} B H_1 H_2 H_3 dq_1 dq_2 dq_3 + \\ &+ \int_b^{q_2} \int_c^{q_3} [G_3(q_1, q_2, q_3, t) H_2(q_1, q_2, q_3) H_3(q_1, q_2, q_3) + \\ &+ G_{-1}(a, q_2, q_3, t) H_2(a, q_2, q_3) H_3(a, q_2, q_3)] dq_2 dq_3 + \\ &+ \int_c^{q_3} \int_a^{q_1} [G_2(q_1, q_2, q_3, t) H_3(q_1, q_2, q_3) H_1(q_1, q_2, q_3) + \\ &+ G_{-2}(q_1, b, q_3, t) H_3(q_1, b, q_3) H_1(q_1, b, q_3)] dq_3 dq_1 + \\ &+ \int_a^{q_1} \int_c^{q_3} [G_3(q_1, q_2, q_3, t) H_1(q_1, q_2, q_3) H_2(q_1, q_2, q_3) + \\ &+ G_{-3}(q_2, q_2, c, t) H_1(q_1, q_2, c) H_2(q_1, q_2, c)] dq_1 dq_2 \quad (3.8) \end{aligned}$$

where G_1, G_2, G_3 denote the surface densities of the flow through the boundaries of a curvilinear parallelepiped with normals

parallel to axes q_1 , q_2 and q_3 , and G_{-1} , G_{-2} , G_{-3} denote the same quantities for contrarily directed normals.

Differentiating both parts of equation (3.8) with respect to q_1 , q_2 and q_3 , we easily arrive at the equation

$$\frac{\partial A}{\partial t} = B + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial(G_1 H_2 H_3)}{\partial q_1} + \frac{\partial(G_2 H_3 H_1)}{\partial q_2} + \frac{\partial(G_3 H_1 H_2)}{\partial q_3} \right] \quad (3.9)$$

Let us resolve each of the magnitudes G_1 , G_2 , G_3 into two; we put

$$G_1 = -v_1 A + C_1, \quad G_2 = -v_2 A + C_2, \quad G_3 = -v_3 A + C_3 \quad (3.10)$$

where v_1 , v_2 , and v_3 are the projections of the velocity vector \mathbf{v} of the medium upon axes q_1 , q_2 and q_3 .

The meaning of the components in formulas (3.10) is altogether clear. If the moving medium were being transferred like an ordinary (not gaseous) continuous medium and if the flow of magnitude Φ were associated only with the macroscopic movement of the substance through surface S , then we would have only the first components in formulas (3.10). In actual fact, on account of the molecular composition of real media, the flow through the surface may be associated not only with the macroscopic motion, but also with the molecular motions inside the substance which is moving with velocity \mathbf{v} . For this reason corrections to the first components are imperative. These corrections, designated as C_1 , C_2 and C_3 , are none other than the flows through an area

which is moving with the velocity of the medium. Inserting (3.10)

into (3.9), we obtain

$$\begin{aligned} & \frac{\partial A}{\partial t} + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (A v_1 H_2 H_3) + \frac{\partial}{\partial q_2} (A v_2 H_3 H_1) + \frac{\partial}{\partial q_3} (A v_3 H_1 H_2) \right] = \\ & = B + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (C_1 H_2 H_3) + \frac{\partial}{\partial q_2} (C_2 H_3 H_1) + \frac{\partial}{\partial q_3} (C_3 H_1 H_2) \right] \end{aligned} \quad (3.11)$$

Or again

$$\begin{aligned} & \frac{\partial A}{\partial t} + \frac{v_1}{H_1} \frac{\partial A}{\partial q_1} + \frac{v_2}{H_2} \frac{\partial A}{\partial q_2} + \frac{v_3}{H_3} \frac{\partial A}{\partial q_3} + \\ & + \frac{A}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (v_1 H_2 H_3) + \frac{\partial}{\partial q_2} (v_2 H_3 H_1) + \frac{\partial}{\partial q_3} (v_3 H_1 H_2) \right] = \\ & = B + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (C_1 H_2 H_3) + \frac{\partial}{\partial q_2} (C_2 H_3 H_1) + \frac{\partial}{\partial q_3} (C_3 H_1 H_2) \right] \end{aligned} \quad (3.12)$$

If we take advantage of the well-known formulas

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial t} + \frac{v_1}{H_1} \frac{\partial A}{\partial q_1} + \frac{v_2}{H_2} \frac{\partial A}{\partial q_2} + \frac{v_3}{H_3} \frac{\partial A}{\partial q_3} \\ \operatorname{div} \mathbf{v} &= \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (v_1 H_2 H_3) + \frac{\partial}{\partial q_2} (v_2 H_3 H_1) + \frac{\partial}{\partial q_3} (v_3 H_1 H_2) \right] \end{aligned} \quad (3.13)$$

then equation (3.12) will be rewritten in the following form:

$$\frac{dA}{dt} + A \operatorname{div} \mathbf{v} = B + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (C_1 H_2 H_3) + \frac{\partial}{\partial q_2} (C_2 H_3 H_1) + \frac{\partial}{\partial q_3} (C_3 H_1 H_2) \right] \quad (3.14)$$

(Equation (3.14) may also be obtained by consideration of a moving volume V , bounded by surface S , the points of which are moving with the macroscopic velocity \mathbf{v} of the motion of the medium.

In this case, instead of (3.6) the following equation is obtained:

$$\iiint_V \left(\frac{\partial A}{\partial t} + \operatorname{div} A \mathbf{v} \right) dV = \iiint_V B dV + \iint_S C_n dS \quad (3.6a)$$

where C_n is the flow of the quantity Φ through an area which is moving with the gas and which has the normal \mathbf{n} . From (3.6a) follows (3.14), and simultaneously the meaning of magnitudes C_1, C_2, C_3 is revealed again, even more graphically.)

This equation [3.14] represents the desired expression in differential form of the general law of the variation of quantity A in accordance with the assumptions made above concerning the factors which determine the variation of quantity Φ , which is related to A by formula (3.1)

4. EQUATIONS OF MOTION OF A MEDIUM

With the aid of equation (3.14) the equations of motion of a medium for the calculation of the flows of mass, momentum

and energy will be derived very simply, if implied in the laws of variation which are expressed by equation (3.14) are the laws of conservation of mass, momentum, and conservation of energy.

In order to obtain the equations of continuity, it is necessary to take the law of conservation of mass and, assuming the absence of spatially distributed sources, to put

$$\Phi = M, A = \rho, B = 0, C_1 = Q_1, C_2 = Q_2, C_3 = Q_3 \quad (4.1)$$

where M is the mass, ρ is the density, Q_1, Q_2, Q_3 are flows of mass in consequence of self-diffusion through regions perpendicular to the coordinate axes.

Then we obtain the equation

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} = \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (Q_1 H_2 H_3) + \frac{\partial}{\partial q_2} (Q_2 H_1 H_3) + \frac{\partial}{\partial q_3} (Q_3 H_1 H_2) \right] \quad (4.2)$$

In order to obtain the equations of motion correctly, it is necessary to follow the law of momentum and put

$$\Phi = \mathbf{K}, A = \rho \mathbf{v}, B = \rho \mathbf{F}, C_1 = \boldsymbol{\tau}_1, C_2 = \boldsymbol{\tau}_2, C_3 = \boldsymbol{\tau}_3 \quad (4.3)$$

where \mathbf{K} is the momentum, \mathbf{v} is the velocity, \mathbf{F} is the mass force, $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \boldsymbol{\tau}_3$ are the flows of momentum through regions perpendicular to the axes, or, what is the same thing, the stresses of the surface forces.

Substituting (4.3) in (3.14) we obtain

$$\frac{d}{dt}(\rho \mathbf{v}) + \rho \mathbf{v} \operatorname{div} \mathbf{v} = \quad (4.4)$$

$$= \rho \mathbf{F} + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (\tau_1 H_2 H_3) + \frac{\partial}{\partial q_2} (\tau_2 H_3 H_1) + \frac{\partial}{\partial q_3} (\tau_3 H_1 H_2) \right]$$

or

$$\rho \frac{d\mathbf{v}}{dt} + \mathbf{v} \left(\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} \right) = \quad (4.5)$$

$$= \rho \mathbf{F} + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (\tau_1 H_2 H_3) + \frac{\partial}{\partial q_2} (\tau_2 H_3 H_1) + \frac{\partial}{\partial q_3} (\tau_3 H_1 H_2) \right]$$

Substituting in the parenthetical expression in the left hand side of (4.5) with the aid of (4.2), we obtain

$$\begin{aligned} & \rho \frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (Q_1 H_2 H_3) + \frac{\partial}{\partial q_2} (Q_2 H_3 H_1) + \frac{\partial}{\partial q_3} (Q_3 H_1 H_2) \right] = \\ & = \rho \mathbf{F} + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (\tau_1 H_2 H_3) + \frac{\partial}{\partial q_2} (\tau_2 H_3 H_1) + \frac{\partial}{\partial q_3} (\tau_3 H_1 H_2) \right] \quad (4.6) \end{aligned}$$

or

$$\rho \frac{d\mathbf{v}}{dt} + \frac{\mathbf{v}}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (Q_1 H_2 H_3) + \frac{\partial}{\partial q_2} (Q_2 H_1 H_3) + \frac{\partial}{\partial q_3} (Q_3 H_1 H_2) \right] = \rho \mathbf{F} + \frac{1}{H_1} \frac{\partial \tau_1}{\partial q_1} + \frac{1}{H_2} \frac{\partial \tau_2}{\partial q_2} + \frac{1}{H_3} \frac{\partial \tau_3}{\partial q_3} + \frac{\tau_1}{H_1 H_2 H_3} \frac{\partial}{\partial q_1} (H_2 H_3) + \frac{\tau_2}{H_1 H_2 H_3} \frac{\partial}{\partial q_2} (H_1 H_3) + \frac{\tau_3}{H_1 H_2 H_3} \frac{\partial}{\partial q_3} (H_1 H_2) \quad (4.7)$$

Practically, one is obliged to use equation (4.7) not in vector form, but in projections onto the curvilinear coordinate axes. It is therefore necessary to make projections of the vectors which enter into this equation onto the curvilinear coordinate axes.

The formulas for the projections of an acceleration $d\mathbf{w}/dt$ are well known:

$$W_i = \left(\frac{d\mathbf{w}}{dt} \right)_i = \frac{\partial v_i}{\partial t} + \frac{v_1}{H_2} \frac{\partial v_i}{\partial q_1} + \frac{v_2}{H_2} \frac{\partial v_i}{\partial q_2} + \frac{v_3}{H_3} \frac{\partial v_i}{\partial q_3} + \frac{v_1 v_{i+1}}{H_1 H_{i+1}} \frac{\partial H_i}{\partial q_{i+1}} + \frac{v_1 v_{i+2}}{H_1 H_{i+2}} \frac{\partial H_i}{\partial q_{i+2}} - \frac{v_{i+1}^2}{H_1 H_{i+1}} \frac{\partial H_{i+1}}{\partial q_i} - \frac{v_{i+2}^2}{H_1 H_{i+2}} \frac{\partial H_{i+2}}{\partial q_i} \quad (i=1,2,3) \quad (4.8)$$

where $v_i = v_{i+3}$ are the projections of the velocity onto the

curvilinear coordinate axes, and $H_1 = H_1 + 3$ are the coefficients of Lamé.

The derivation of formulas for the projections of the derivatives with respect to the coordinates, of vectors \vec{r}_1 , \vec{r}_2 , and \vec{r}_3 also presents no difficulty. We put

$$\frac{\partial \vec{r}_k}{\partial q_l} = B_{kl}^{(1)} \vec{e}_1 + B_{kl}^{(2)} \vec{e}_2 + B_{kl}^{(3)} \vec{e}_3 \quad (k=1,2,3; l=1,2,3) \quad (4.9)$$

where \vec{e}_1 , \vec{e}_2 and \vec{e}_3 are the "orths" [unit vectors] of the curvilinear coordinates, and $B_{kl}^{(m)}$ are coefficients which we shall determine later.

Taking (4.9) into consideration, we shall have, for the derivative of any vector \vec{a} with respect to the coordinate q_l , the formula

$$\frac{\partial \vec{a}}{\partial q_l} = \sum_{j=1}^3 \left(\frac{\partial a_j}{\partial q_l} + a_1 B_{1l}^{(j)} + a_2 B_{2l}^{(j)} + a_3 B_{3l}^{(j)} \right) \vec{e}_j \quad (4.10)$$

where a_1, a_2, a_3 are the projections of the vector onto the curvilinear axes.

Using formulas (4.8) and (4.10), we obtain from the vector equation (4.7) three scalar equations:

$$\begin{aligned}
 & \rho \left[\frac{\partial v_j}{\partial t} + \frac{v_1}{H_1} \frac{\partial v_j}{\partial q_1} + \frac{v_2}{H_2} \frac{\partial v_j}{\partial q_2} + \frac{v_3}{H_3} \frac{\partial v_j}{\partial q_3} + \right. \\
 & \left. + \frac{v_j v_{j+1}}{H_j H_{j+1}} \frac{\partial H_j}{\partial q_{j+1}} + \frac{v_j v_{j+2}}{H_j H_{j+2}} \frac{\partial H_j}{\partial q_{j+2}} - \frac{v_{j+1}^2}{H_j H_{j+1}} \frac{\partial H_{j+1}}{\partial q_j} - \frac{v_{j+2}^2}{H_j H_{j+2}} \frac{\partial H_{j+2}}{\partial q_j} \right] + \\
 & + \frac{v_j}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (Q_1 H_2 H_3) + \frac{\partial}{\partial q_2} (Q_2 H_3 H_1) + \frac{\partial}{\partial q_3} (Q_3 H_1 H_2) \right] = \\
 & = \rho F_j + \frac{\tau_{1j}}{H_1 H_2 H_3} \frac{\partial}{\partial q_1} (H_2 H_3) + \frac{\tau_{2j}}{H_1 H_2 H_3} \frac{\partial}{\partial q_2} (H_3 H_1) + \frac{\tau_{3j}}{H_1 H_2 H_3} \frac{\partial}{\partial q_3} (H_1 H_2) + \\
 & + \frac{1}{H_1} \left(\frac{\partial \tau_{11}}{\partial q_1} + \tau_{11} B_{11}^{(1)} + \tau_{12} B_{21}^{(1)} + \tau_{13} B_{31}^{(1)} \right) + \\
 & + \frac{1}{H_2} \left(\frac{\partial \tau_{2j}}{\partial q_2} + \tau_{21} B_{12}^{(j)} + \tau_{22} B_{22}^{(j)} + \tau_{23} B_{32}^{(j)} \right) + \\
 & + \frac{1}{H_3} \left(\frac{\partial \tau_{3j}}{\partial q_3} + \tau_{31} B_{13}^{(j)} + \tau_{32} B_{23}^{(j)} + \tau_{33} B_{33}^{(j)} \right) \quad (j=1,2,3) \quad (4.11)
 \end{aligned}$$

Let us find the expressions for the coefficients $B_{kl}^{(m)}$. From (4.9) we have

$$B_{xl}^{(m)} = \frac{\partial i_k}{\partial q_l} \cdot i_m \quad (4.12)$$

If \mathbf{i} , \mathbf{j} , \mathbf{k} are the unit vectors of the Cartesian coordinate axes, and if the relation between the Cartesian and curvilinear coordinates is given by the formulas

$$x = x(q_1, q_2, q_3), \quad y = y(q_1, q_2, q_3), \quad z = z(q_1, q_2, q_3) \quad (4.13)$$

then, obviously, we have

$$\mathbf{i}_k = \frac{1}{H_k} \left(\frac{\partial x}{\partial q_k} \mathbf{i} + \frac{\partial y}{\partial q_k} \mathbf{j} + \frac{\partial z}{\partial q_k} \mathbf{k} \right)$$

$$\mathbf{i}_m = \frac{1}{H_m} \left(\frac{\partial x}{\partial q_m} \mathbf{i} + \frac{\partial y}{\partial q_m} \mathbf{j} + \frac{\partial z}{\partial q_m} \mathbf{k} \right) \quad (4.14)$$

$$\frac{\partial \mathbf{i}_k}{\partial q_l} = \frac{\partial}{\partial q_l} \left(\frac{1}{H_k} \frac{\partial x}{\partial q_k} \right) \mathbf{i} + \frac{\partial}{\partial q_l} \left(\frac{1}{H_k} \frac{\partial y}{\partial q_k} \right) \mathbf{j} + \frac{\partial}{\partial q_l} \left(\frac{1}{H_k} \frac{\partial z}{\partial q_k} \right) \mathbf{k}$$

Consequently

$$\mathcal{B}_{kl}^{(m)} = \frac{1}{H_m} \left[\frac{\partial x}{\partial q_m} \frac{\partial}{\partial q_l} \left(\frac{1}{H_k} \frac{\partial x}{\partial q_k} \right) + \frac{\partial y}{\partial q_m} \frac{\partial}{\partial q_l} \left(\frac{1}{H_k} \frac{\partial y}{\partial q_k} \right) + \frac{\partial z}{\partial q_m} \frac{\partial}{\partial q_l} \left(\frac{1}{H_k} \frac{\partial z}{\partial q_k} \right) \right] \quad (4.15)$$

Equations (4.11) together with (4.15) are the equations we were seeking, expressing in differential form the law of momenta.

Let us pass on to the derivation of the equation of energy. For this let us make use of the law of conservation of energy.

We put

$$\Phi = U^*, \quad A = \frac{1}{2} \rho v^2 + \rho E, \quad B = \rho \mathbf{F} \cdot \mathbf{v} + \epsilon \quad (4.16)$$

$$C_1 = \boldsymbol{\tau}_1 \cdot \mathbf{v} + t_1, \quad C_2 = \boldsymbol{\tau}_2 \cdot \mathbf{v} + t_2, \quad C_3 = \boldsymbol{\tau}_3 \cdot \mathbf{v} + t_3$$

[sic]

where U^* is the total energy; E is the internal energy of a unit mass; $\rho \mathbf{F} \cdot \mathbf{v}$ is the power evolved by the volume forces; $\boldsymbol{\tau}_1 \cdot \mathbf{v}$, $\boldsymbol{\tau}_2 \cdot \mathbf{v}$, $\boldsymbol{\tau}_3 \cdot \mathbf{v}$ are the powers evolved by the surface forces on a unit area of regions perpendicular to the coordinate axes; ϵ is the volume velocity [specific rate] of liberation of chemical, light, etc. energy; t_1 , t_2 , t_3 are flows of heat through areas perpendicular to the axes.

Substituting (4.16) in (3.14), we obtain

$$\begin{aligned} & \frac{d}{dt} \left[\rho \left(\frac{v^2}{2} + E \right) \right] + \rho \left(\frac{v^2}{2} + E \right) \operatorname{div} \mathbf{v} = \rho \mathbf{F} \cdot \mathbf{v} + \epsilon + \\ & + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (\boldsymbol{\tau}_1 \cdot \mathbf{v} H_2 H_3) + \frac{\partial}{\partial q_2} (\boldsymbol{\tau}_2 \cdot \mathbf{v} H_3 H_1) + \frac{\partial}{\partial q_3} (\boldsymbol{\tau}_3 \cdot \mathbf{v} H_1 H_2) \right] + \\ & + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (t_1 H_2 H_3) + \frac{\partial}{\partial q_2} (t_2 H_3 H_1) + \frac{\partial}{\partial q_3} (t_3 H_1 H_2) \right] \quad (4.17) \end{aligned}$$

or

$$\rho \frac{dE}{dt} + \left(E + \frac{v^2}{2}\right) \left(\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v}\right) + \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = \rho \mathbf{F} \cdot \mathbf{v} + \epsilon \quad (4.18)$$

$$+ \frac{\mathbf{v}}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (\tau_1 H_2 H_3) + \frac{\partial}{\partial q_2} (\tau_2 H_3 H_1) + \frac{\partial}{\partial q_3} (\tau_3 H_1 H_2) \right] + \frac{\tau_1}{H_1} \frac{\partial \mathbf{v}}{\partial q_1} + \frac{\tau_2}{H_2} \frac{\partial \mathbf{v}}{\partial q_2} + \frac{\tau_3}{H_3} \frac{\partial \mathbf{v}}{\partial q_3} + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (t_1 H_2 H_3) + \frac{\partial}{\partial q_2} (t_2 H_3 H_1) + \frac{\partial}{\partial q_3} (t_3 H_1 H_2) \right]$$

Using equation (4.2) and equation (4.6) scalar-multiplied by \mathbf{v} , we can produce obvious simplifications in the foregoing equation.

Then we obtain

$$\begin{aligned} \rho \frac{dE}{dt} + \left(E - \frac{v^2}{2}\right) \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (Q_1 H_2 H_3) + \frac{\partial}{\partial q_2} (Q_2 H_3 H_1) + \frac{\partial}{\partial q_3} (Q_3 H_1 H_2) \right] = \\ = \epsilon + \frac{\tau_1}{H_1} \cdot \frac{\partial \mathbf{v}}{\partial q_1} + \frac{\tau_2}{H_2} \cdot \frac{\partial \mathbf{v}}{\partial q_2} + \frac{\tau_3}{H_3} \cdot \frac{\partial \mathbf{v}}{\partial q_3} + \\ + \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (t_1 H_2 H_3) + \frac{\partial}{\partial q_2} (t_2 H_3 H_1) + \frac{\partial}{\partial q_3} (t_3 H_1 H_2) \right] \quad (4.19) \end{aligned}$$

This is the desired equation of energy. In writing it in expanded form we should keep in mind that the derivatives of the velocity vector with respect to the coordinates must be calculated with the aid of formula (4.10).

In the case of Cartesian coordinates x , y and z , when $H_1 = H_2 = H_3 = 1$, and $B_{kl}^{(m)} = 0$, all the revealed equations are substantially simplified, and from equations (4.2), (4.11) and (4.18) the following simple equations are obtained:

$$\frac{dp}{dt} + \rho \operatorname{div} v = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z}$$

$$\rho \frac{dv_x}{dt} + v_x \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \right) = \rho F_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

$$\rho \frac{dv_y}{dt} + v_y \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \right) = \rho F_y + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \quad (4.20)$$

$$\rho \frac{dv_z}{dt} + v_z \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \right) = \rho F_z + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

$$\rho \frac{dE}{dt} + \left(E - \frac{v^2}{2} \right) \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial Q_z}{\partial z} \right) =$$

$$= \epsilon + \frac{\partial t_x}{\partial x} + \frac{\partial t_y}{\partial y} + \frac{\partial t_z}{\partial z} + \tau_x \cdot \frac{\partial v}{\partial x} + \tau_y \cdot \frac{\partial v}{\partial y} + \tau_z \cdot \frac{\partial v}{\partial z}$$

The quantities Q_i , τ_i and t_i which enter into the obtained equations are determined in the following paragraph.

5. EXPRESSIONS FOR FLOWS OF MASS, MOMENTUM AND HEAT

For the determination of the magnitudes of the flows of mass Q_1 , momentum τ_{ik} and heat t_1 through a region which is moving with the velocity of the gas w perpendicularly to axes q_1 (In order to simplify the exposition, the indication that the question is of flows through moving regions is dropped), we shall assume that these magnitudes must be linear functions of the first derivatives of the hydrodynamic parameters with respect to Cartesian coordinates. (This assumption may be accepted on the basis of numerous experiments devoted to the study of various types of particular cases of the phenomena of transfer of mass, momentum and thermal energy in gases. The results of these experiments show that with adherence to certain conditions the flows of these quantities actually do turn out to be linear functions of the first derivatives of the hydrodynamic parameters v_x, v_y, v_z, ρ and T with respect to the Cartesian coordinates x, y and z .)

It is evident that in accepting such an assumption we somewhat narrow the class of gas motions which are accessible to research through the aid of the derived equations.

In fact, in justice to this assumption, it is, for example, completely imperative that the hydrodynamic parameters should, with sufficient accuracy, permit linear approximation at distances of the order of several times the mean free path of the molecules, since otherwise the state of a gas in a volume which is forming flows and which has, for all practical purposes,

dimensions on the order of several times the mean free path of the molecules, would not be determined by the assignment of the hydrodynamic parameters themselves and of their first derivatives at some point of the volume. Not having any data on the state of a gas in a volume which is forming currents, it would actually be necessary to state the problem of the search for an expression for these flows in terms of the hydrodynamic parameters and their first derivatives. At any rate, in spite of a certain narrowing of the class of gas motions accessible to such an investigation, experiment and the kinetic theory of gases affirm that the basic assumption formulated above will with sufficient accuracy be fulfilled in a very broad class of gas motions of practical interest, and the basic assumptions which were formulated at the beginning of this work suffice to assure its correctness.

Here the expressions for the currents are first determined in Cartesian coordinates, and then in general orthogonal curvilinear coordinates.) The coefficients of these linear functions must depend only on the hydrodynamic parameters themselves. It is obvious that they cannot depend on the projections v_x , v_y and v_z and the velocity \mathbf{v} . Therefore they must be functions of density and temperature only. (The question is of flows through a region which is moving with the gas. The velocity of the gas relative to the region is always zero. Therefore the values of magnitudes v_x , v_y and v_z cannot be reflected in the flows.)

Let us set about the search for linear forms for Q_x , Q_y and Q_z . Proceeding from what has been said, we must put for Q_x

$$\begin{aligned}
Q_x = & A_x + a_{1x} \frac{\partial \rho}{\partial x} + a_{2x} \frac{\partial \rho}{\partial y} + a_{3x} \frac{\partial \rho}{\partial z} + \\
& + b_{1x} \frac{\partial T}{\partial x} + b_{2x} \frac{\partial T}{\partial y} + b_{3x} \frac{\partial T}{\partial z} + \\
& + c_{1x} \frac{\partial v_x}{\partial x} + c_{2x} \frac{\partial v_x}{\partial y} + c_{3x} \frac{\partial v_x}{\partial z} + \quad (5.1) \\
& + d_{1x} \frac{\partial v_y}{\partial x} + d_{2x} \frac{\partial v_y}{\partial y} + d_{3x} \frac{\partial v_y}{\partial z} + \\
& + e_{1x} \frac{\partial v_z}{\partial x} + e_{2x} \frac{\partial v_z}{\partial y} + e_{3x} \frac{\partial v_z}{\partial z}
\end{aligned}$$

It is evident that with a reversal of the direction of the x axis, the sign of magnitude Q_x must change. Therefore, in the right hand side of formula (5.1) there can be no term which does not change sign with the reversal of the direction of the x axis. Hence

$$A_x = a_{2x} = a_{3x} = b_{2x} = b_{3x} = c_{1x} = d_{2x} = d_{3x} = e_{2x} = e_{3x} = 0 \quad (5.2)$$

Further, Q_x must not depend upon the directions of the y and z axes. Therefore

$$c_{2x} = c_{3x} = d_{1x} = e_{1x} = 0 \quad (5.3)$$

Consequently

$$Q_x = a_{1x} \frac{\partial p}{\partial x} + b_{1x} \frac{\partial T}{\partial x} \quad (5.4)$$

Since the x, y and z axes possess altogether symmetric properties, then it must be that

$$a_{1x} = a_{2y} = a_{3z} = D_1, \quad b_{1x} = b_{2y} = b_{3z} = D_2 \quad (5.5)$$

Thus we have

$$Q_x = D_1 \frac{\partial p}{\partial x} + D_2 \frac{\partial T}{\partial x}, \quad Q_y = D_1 \frac{\partial p}{\partial y} + D_2 \frac{\partial T}{\partial y}, \quad Q_z = D_1 \frac{\partial p}{\partial z} + D_2 \frac{\partial T}{\partial z} \quad (5.6)$$

We shall call coefficient D_1 the coefficient of density self-diffusion, and coefficient D_2 the coefficient of temperature self-diffusion.

For the linear forms which give the flows of heat t_x , t_y and t_z , repeating the expressions just written, we arrive at the formulas

$$t_x = K_1 \frac{\partial p}{\partial x} + K_2 \frac{\partial T}{\partial x}, \quad t_y = K_1 \frac{\partial p}{\partial y} + K_2 \frac{\partial T}{\partial y}, \quad t_z = K_1 \frac{\partial p}{\partial z} + K_2 \frac{\partial T}{\partial z} \quad (5.7)$$

where K_1 may be called the coefficient of density heat transfer, and K_2 the coefficient of temperature heat transfer. Coefficients

D_1, D_2 and K_1, K_2 are encountered in the kinetic theory of gases.

Finally, let us establish, for Cartesian coordinates, the appearance of the linear forms which give the projections of vectors τ_x, τ_y and τ_z on the x, y and z axes.

Let us return to equation (3.14) and write it in Cartesian coordinates in conformity with the law of moments of momentum.

Let us put

$$\begin{aligned} \Phi &= \mathbf{L}, \quad \mathbf{A} = \mathbf{r} \times \rho \mathbf{v}, \quad \mathbf{B} = \mathbf{r} \times \rho \mathbf{F} \\ C_1 &= \mathbf{r} \times \tau_x, \quad C_2 = \mathbf{r} \times \tau_y, \quad C_3 = \mathbf{r} \times \tau_z \end{aligned} \quad (5.8)$$

where \mathbf{L} is the moment of momentum, \mathbf{r} is the radius vector of a moving point of the medium, $\mathbf{r} \times \rho \mathbf{F}$ is the moment of the mass forces on a unit volume, $\mathbf{r} \times \tau_x, \mathbf{r} \times \tau_y, \mathbf{r} \times \tau_z$ are flows of the moment of momentum through moving areas perpendicular to the axes.

Introducing (5.8) into (3.14), we obtain

$$\begin{aligned} \frac{d}{dt} (\mathbf{r} \times \rho \mathbf{v}) + (\mathbf{r} \times \rho \mathbf{v}) \operatorname{div} \mathbf{v} &= \mathbf{r} \times \rho \mathbf{F} + \frac{\partial}{\partial x} (\mathbf{r} \times \tau_x) + \\ &+ \frac{\partial}{\partial y} (\mathbf{r} \times \tau_y) + \frac{\partial}{\partial z} (\mathbf{r} \times \tau_z) \end{aligned} \quad (5.9)$$

or

$$\begin{aligned} \mathbf{r} \times \left[\frac{d}{dt} (\rho \mathbf{v}) + \rho \mathbf{v} \operatorname{div} \mathbf{v} - \rho \mathbf{F} - \frac{\partial \tau_x}{\partial x} - \frac{\partial \tau_y}{\partial y} - \frac{\partial \tau_z}{\partial z} \right] + \\ + \frac{d\mathbf{r}}{dt} \times \rho \mathbf{v} + \frac{\partial \mathbf{r}}{\partial x} \times \tau_x + \frac{\partial \mathbf{r}}{\partial y} \times \tau_y + \frac{\partial \mathbf{r}}{\partial z} \times \tau_z = 0 \end{aligned} \quad (5.10)$$

Here the [expression in] square brackets is equal to zero on the basis of (4.4). Moreover

$$\frac{dr}{dt} \times \rho \mathbf{v} = \mathbf{v} \times \rho \mathbf{v} = 0, \quad \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i}, \quad \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j}, \quad \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k} \quad (5.11)$$

Consequently

$$\mathbf{i} \times \tau_x + \mathbf{j} \times \tau_y + \mathbf{k} \times \tau_z = 0 \quad (5.12)$$

This vector equality is equivalent to three scalar equalities

$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{zx} = \tau_{xz}$$

representing an extension of the well-known symmetry characteristic of the stress tensor of a continuous fluid with self-diffusion.

Having this in mind, and in conformity with general considerations, we put

$$\begin{aligned} \tau_{ik} = & A_{ik} + a_{ik}^{(1)} \frac{\partial p}{\partial x} + a_{ik}^{(2)} \frac{\partial p}{\partial y} + a_{ik}^{(3)} \frac{\partial p}{\partial z} + \\ & + b_{ik}^{(1)} \frac{\partial T}{\partial x} + b_{ik}^{(2)} \frac{\partial T}{\partial y} + b_{ik}^{(3)} \frac{\partial T}{\partial z} + \\ & + c_{ik}^{(1)} \frac{\partial v_x}{\partial x} + c_{ik}^{(2)} \frac{\partial v_y}{\partial y} + c_{ik}^{(3)} \frac{\partial v_z}{\partial z} + \end{aligned} \quad (5.13)$$

$$\begin{aligned}
 & + d_{ik}^{(1)} \left(\frac{\partial v_y}{\partial y} + \frac{\partial v_y}{\partial x} \right) + d_{ik}^{(2)} \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) + d_{ik}^{(3)} \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) + (5.13) \text{ (con't)} \\
 & + e_{ik}^{(1)} \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) + e_{ik}^{(2)} \left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y} \right) + e_{ik}^{(3)} \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z} \right)
 \end{aligned}$$

Magnitudes A_{ik} , and the other coefficients in (5.13) as well, do not depend on the values of the derivatives of the hydrodynamic parameters with respect to the coordinates; they may be found in those values which they take in a gas with constant hydrodynamic parameters

$$A_{kk} = -p, \quad A_{ik} = 0 \quad i \neq k \quad (5.14)$$

where p is the pressure in a gas with constant hydrodynamic parameters, which is determined for a perfect gas by Clapeyron's equation

$$p = R\rho T \quad (5.15)$$

Further, since magnitudes τ_{ik} do not change with a reversal in the direction of all the axes, whereas the derivatives of ρ and T with respect to the coordinates do change sign with such a reversal, then

$$a_{ik}^{(1)} = a_{ik}^{(2)} = a_{ik}^{(3)} = b_{ik}^{(1)} = b_{ik}^{(2)} = b_{ik}^{(3)} = 0 \quad (5.16)$$

Finally, in consequence of (5.12), it is evident that we have

$$e_{ik}^{(1)} = e_{ik}^{(2)} = e_{ik}^{(3)} = 0 \quad (5.17)$$

Therefore, magnitudes τ_{ik} (the components of the stress tensor) prove to be linear functions of the components of the rate of strains tensor only. This, as is known [5], suffices for obtaining the equalities

$$\begin{aligned} \tau_{xx} &= -p + \mu_1 \operatorname{div} \mathbf{v} + 2\mu \frac{\partial v_x}{\partial x}, & \tau_{xy} &= \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \\ \tau_{yy} &= -p + \mu_1 \operatorname{div} \mathbf{v} + 2\mu \frac{\partial v_y}{\partial y}, & \tau_{yz} &= \mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \\ \tau_{zz} &= -p + \mu_1 \operatorname{div} \mathbf{v} + 2\mu \frac{\partial v_z}{\partial z}, & \tau_{zx} &= \mu \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \end{aligned} \quad (5.18)$$

where μ_1 and μ are some functions, generally speaking, of the density ρ and temperature T . Quantity μ , as is known, is called the coefficient of viscosity, and quantity μ_1 , the coefficient of second viscosity.

If we introduce into consideration the tensors: of stress \mathbf{T} , of rates of strains \mathbf{II} , and the unit tensor \mathbf{I} , then all the foregoing equalities are incorporated into one:

$$\mathbf{T} = (-p + \mu_1 \operatorname{div} \mathbf{v}) \mathbf{I} + 2\mu \mathbf{II} \quad (5.19)$$

which is extremely convenient for conversion to any orthogonal curvilinear coordinates. Converting in (5.6) and (5.7) to orthogonal curvilinear coordinates, we obviously obtain

$$Q_i = D_1 \frac{1}{H_i} \frac{\partial p}{\partial q_i} + D_2 \frac{1}{H_i} \frac{\partial T}{\partial q_i}, \quad t_i = K_1 \frac{1}{H_i} \frac{\partial p}{\partial q_i} + K_2 \frac{1}{H_i} \frac{\partial T}{\partial q_i} \quad (5.20)$$

Further, converting in (5.19) to curvilinear coordinates, and performing the usual calculations [5], we obtain the last of the desired formulas:

$$\begin{aligned} \tau_{ii} = & -p + \mu_1 \frac{1}{H_1 H_2 H_3} \left[\frac{\partial}{\partial q_1} (v_1 H_2 H_3) + \frac{\partial}{\partial q_2} (v_2 H_1 H_3) + \frac{\partial}{\partial q_3} (v_3 H_1 H_2) \right] + \\ & + 2\mu \left[\frac{1}{H_i} \frac{\partial v_i}{\partial q_i} + \frac{v_{i+1}}{H_i H_{i+1}} \frac{\partial H_i}{\partial q_{i+1}} + \frac{v_{i+2}}{H_i H_{i+2}} \frac{\partial H_i}{\partial q_{i+2}} \right] \quad (5.21) \\ \tau_{ik} = & \mu \left[\frac{1}{H_k} \frac{\partial v_i}{\partial q_k} + \frac{1}{H_i} \frac{\partial v_k}{\partial q_i} - \frac{v_i}{H_i H_k} \frac{\partial H_i}{\partial q_k} - \frac{v_k}{H_i H_k} \frac{\partial H_k}{\partial q_i} \right] \end{aligned}$$

The obtained expressions for the flows of mass, momentum and heat contain six coefficients: $D_1, D_2, K_1, K_2, \mu_1, \mu$.

6. EXPRESSIONS FOR COEFFICIENTS $\mu_1, D_1, D_2, K_1, K_2$.

For the establishment of expressions for coefficients μ_1, D_1, D_2, K_1 and K_2 we shall make use of the methods of the theory of dimensions.

If we introduce into consideration the coefficient c_v of thermal capacity ^[specific heat] for a constant volume, expressed in mechanical rather than thermal units, we shall have the following correlations between the dimensions:

$$[\mu_1] = [\mu], [D_1] = \left[\frac{\mu}{\rho} \right], [D_2] = \left[\frac{\mu}{T} \right], [K_1] = \left[\frac{\mu c_v T}{\rho} \right], \quad (6.1)$$

$$[K_2] = [\mu c_v]$$

Therefore we may put

$$\mu_1 = a\mu, D_1 = \frac{\mu}{\rho} \alpha_1, D_2 = \frac{\mu}{T} \alpha_2, K_1 = \frac{\mu c_v T}{\rho} \beta_1, K_2 = \mu c_v \beta_2 \quad (6.2)$$

where $a, \alpha_1, \alpha_2, \beta_1$ and β_2 will be nondimensional functions of the nondimensional parameters, which determine the condition of the gas in a state of equilibrium, since the formulas for flows prove to be applicable with the same coefficients for all the states of a gas under consideration, including those which are arbitrarily close to the state of equilibrium.

The equilibrium state of a given perfect gas is fully determined with assignment of its density ρ and temperature T . Not a single nondimensional combination of these magnitudes can be constructed. Therefore, for a perfect gas magnitudes $a, \alpha_1,$

α_2 , β_1 and β_2 will be constants depending on the kind of gas, and for this reason coefficients M_1 , D_1 , D_2 , K_1 and K_2 may be considered known with an accuracy up to that of constants a , α_1 , α_2 , β_1 and β_2 if the coefficient of viscosity is known, since the coefficient c_v of thermal capacity, with volume constant, is constant for a perfect gas.

7. COEFFICIENTS a , α_1 , α_2 , β_1 and β_2

The kinetic theory of gases permits us to expect that the numerical coefficients a , α_1 , α_2 , β_1 and β_2 will be magnitudes of the order of unity. Generally speaking, these magnitudes must be determined through suitably conducted experiments.

At the present time we do not have experimental values of all these coefficients at our disposal, but nevertheless, supported by certain experimental results and by deliberations issuing from the kinetic theory of gases, we shall give numerical values to these coefficients for monoatomic gases.

To begin with, it is well known from the kinetic theory of gases that for a monoatomic gas it must be [6] that

$$a = \frac{2}{3} \quad (7.1)$$

Further, in experiments on ~~self~~ self-diffusion in gases it has been established [7] that one coefficient is the same

for all monoatomic gases (within the limits of experimental accuracy), namely

$$\alpha_1 = 1.30 \quad (7.2)$$

which is also sufficiently close to theoretical values of this coefficient obtained for various molecular models. Having established the value of α_1 , it is not difficult to establish at once the value of β_1 as well.

By the very meaning of coefficient D_1 one may assert that with a constant temperature T and a variable density ρ through a region of area dS with the normal \mathbf{n} during time dt , there will be transferred, as a result of density self-diffusion, a mass Δm , given by the formula

$$\Delta m = D_1 \frac{\partial \rho}{\partial n} dS dt \quad (7.3)$$

This mass possesses the thermal energy Δq , where

$$\Delta q = \Delta m c_v T = c_v T D_1 \frac{\partial \rho}{\partial n} dS dt \quad (7.4)$$

This thermal energy is the thermal energy which passes through our region during time dt as a consequence of density heat transfer. Consequently there is another expression for magnitude Δq

$$\Delta q = K_1 \frac{\partial p}{\partial n} ds dt. \quad (7.5)$$

Equating (7.4) with (7.5), we obtain $\alpha_2 = \beta_1$. Consequently, for monoatomic gases

$$\beta_1 = 1.30 \quad (7.6)$$

Let us turn, further, to the steady transmission of heat through a flat layer of gas, which is bounded by two partitions located at a distance Δl , with a difference ΔT between their temperatures. If we designate the heat flow as q , then from experimental data we easily find the value of magnitude f , which is determined by the formula

$$f = \frac{q}{\mu c_p \Delta T / \Delta l} \quad (7.7)$$

The average experimental value of this magnitude for monoatomic gases [8] is equal to 2.51. On the other hand, it will be demonstrated below (sections 9, 10) that

$$f = \beta_2 - \beta_1 \quad (7.8)$$

Consequently, for monoatomic gases

$$\beta_2 = 3.81. \quad (7.9)$$

Let us now concern ourselves with the search for α_2 .

Let us consider the heat transfer in a quiescent gas of constant density and variable temperature. We designate as Δq the quantity of thermal energy which passes through a region of area dS with the normal \mathbf{n} during the time interval dt . We have, obviously

$$\Delta q = K_2 \frac{\partial T}{\partial n} dS dt. \quad (7.10)$$

Magnitude Δq is made up of two components: Δq_1 and Δq_2 .

The first component, Δq_1 , represents the quantity of thermal energy flowing through the region on account of the temperature variability only, disregarding thermal self-diffusion.

This magnitude may be calculated according to the theoretical formula

$$\Delta q_1 = f^* \mu c_v \frac{\partial T}{\partial n} dS dt. \quad (7.11)$$

Serving as a basis for confidence in applying this theoretical formula to monoatomic gases is the circumstance that for the most diverse models of the molecules of monoatomic gases, extremely close values for the numerical coefficient are obtained.

[9]. Namely, for all theoretically analyzed models of mono-atomic molecules, f^* is found to lie within the range 2.50 to 2.52. Therefore we shall accept

$$\Delta q_1 = 2.51 \mu c_v \frac{\partial T}{\partial n} dS dt \quad (7.12)$$

The second component, Δq_2 , represents the quantity of thermal energy which is transferred by the thermal-self-diffusional flow of mass. If we designate as Δm the mass which passes through the region as a consequence of thermal self-diffusion, then we shall have

$$\Delta m = D_2 \frac{\partial T}{\partial n} dS dt. \quad (7.13)$$

This mass possesses the thermal energy Δq_2 , where

$$\Delta q_2 = \Delta m c_v T = c_v T D_2 \frac{\partial T}{\partial n} dS dt. \quad (7.14)$$

Equating the two expressions obtained for Δq , we get

$$\alpha_2 = \beta_2 - f^* \quad (7.15)$$

Consequently

$$\alpha_2 = 1.30$$

In conformity with our earlier expectations, all the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ and a are found to be magnitudes on the order of unity.

Remark. The relative magnitude of various terms in the equations of motion will differ under different conditions. One may indicate conditions of gas motion under which the self-diffusional terms will have the basic significance, and one may indicate conditions of motion under which only the terms associated with the stress tensor will have basic significance, etc. Therefore, it is impossible to speak of the relative magnitude of various terms in the equations of motion without having specified the class of motion.

An extremely important class of motion from the practical point of view is constituted by motions in which the magnitude of each of the hydrodynamic parameters (velocity \mathbf{v} , density ρ and temperature T) varies by a magnitude of their own order at distances of the order of the same length L .

To such motions belong, for example, motions in a boundary layer under high velocity conditions. In these motions, at points separated from one another by distances of the order of the thickness of the boundary layer δ , the hydrodynamic parameters differ, generally speaking, by magnitudes of the order of the hydrodynamic parameters themselves.

If we have in view just such motions with one characteristic interval for all the hydrodynamic parameters, then the standard conversion to nondimensional magnitudes leads us at once to the

conclusion that in these motions all the self-diffusion, heat transfer and viscosity terms have the same relative magnitude, provided the constants α , α_1 , α_2 , β_1 and β_2 are of one order.

Since according to the kinetic theory of gases all these constants are of the order of unity, it follows from what has been said that for the motions under consideration the retention in the equations of even one term associated with phenomena of transfer makes requisite the retention of all the other terms associated with these phenomena.

This means, moreover, that for motions of the class in question it is reasonable to employ either completely developed equations, or equations of motion of an ideally compressible fluid.

8. BOUNDARY CONDITIONS FOR THE SYSTEM OF DIFFERENTIAL EQUATIONS OF GAS MOTION

Required for the integration of the obtained system of differential equations of gas motion are the boundary conditions on the surfaces of solid bodies surrounded by a steady flow of gas and the boundary conditions at infinity, if the area occupied by gas extends to infinity.

(Since the order of the system we have obtained is higher by one than ^{the} the order of the system which is obtained with an incomplete accounting for the phenomena of transfer, the old boundary conditions do not fit the statement of the problem,

if only because their number does not correspond to the new order of the system of equations of motion.)

The problem of the boundary conditions at infinity is very easily solved. It is obvious that these conditions must be implied with the assignment of values to the density, temperature and velocity components at infinity.

The solution of the problem of boundary conditions at the surfaces of streamlined solid bodies is more complex.

First of all it is altogether clear that no mass of gas permeates the surface of a streamlined solid body.

If we designate as n the normal to the surface S of a streamlined body and suppose that the streamlined body is motionless in space, then this physical fact may be written in the following form:

$$\left(D_1 \frac{\partial \rho}{\partial n} + D_2 \frac{\partial T}{\partial n} \right)_S = (\rho v_n)_S \quad (8.1)$$

In order to obtain the boundary conditions at the surface of a streamlined solid body, let us assume that immediately at the surface the gas either moves very slowly or is at rest.

(One may bring forward certain physical deliberations as a basis for this assumption. We may assume a visually smooth streamlined surface with a gas microscopically moving in its vicinity; the gas will be subject to conditions which approach

the conditions of flow through an extremely fine-pored medium.

The tremendous coefficients of resistance to the motion of a gas through various kinds of fine gratings and porous media are well known, as are the utterly insignificant expenditures of gas in motion through gratings and porous media, which with sufficient fineness of the pores are found to be, for practical purposes, zero.)

Disregarding the small velocities of the gas in the directions tangent to the surface S , and calling \mathbf{l}_1 and \mathbf{l}_2 vectors tangent to surface S , we obtain two more marginal conditions:

$$(v_{l_1})_S = 0, \quad (v_{l_2})_S = 0. \quad (8.2)$$

In a quiescent gas we must have a static pressure distribution. Therefore, by a well known approximation, we may put

$$\left(\frac{\partial p}{\partial n}\right)_S = (\rho F_n)_S \quad (8.3)$$

where F_n is the projection of the mass force onto the normal to the surface.

Marginal conditions (8.2) are fairly well confirmed experimentally. Marginal conditions (8.3) must still be experimentally verified and at present may be considered to constitute a likely hypothesis.

9. EXAMPLE OF INTEGRATION
OF ^A ~~THE~~ SYSTEM OF EQUATIONS OF GAS MOTION

Let us consider the very simple one-dimensional problem of steady transmission of heat through a layer of gas situated between two parallel planes, with the condition that the difference between the temperatures of the two partitions is small, and mass forces are absent.

Grant that we have two parallel planes $x = -l$ and $x = +l$. Let the temperatures of these planes be T_1 and T_2 respectively. Let there be such a quantity of gas between the two planes as would have density ρ_0 if the density were constant throughout the entire space occupied by gas. Finally, let the quantity

$$\epsilon^* = \frac{T_2 - T_1}{T_2 + T_1} \quad (9.1)$$

be so small that its square is negligible in comparison with unity.

The considered problem fits the following somewhat modified system of equations of gas motion:

$$\begin{aligned} \frac{d}{dx}(\rho v) &= \frac{d}{dx} \left(D_1 \frac{dp}{dx} + D_2 \frac{dT}{dx} \right) \\ \frac{d}{dx}(\rho v^2) &= \frac{d}{dx} \left(-R_p T + \frac{4}{3} \mu \frac{dv}{dx} \right) \end{aligned} \quad (9.2)$$

$$\rho c_v v \frac{dT}{dx} + \left(c_v T - \frac{v^2}{2} \right) \frac{d}{dx} (\rho v) =$$

(9.2) (cont)

$$= \frac{d}{dx} \left(K_1 \frac{dp}{dx} + K_2 \frac{dT}{dx} \right) + \left(-R_p T + \frac{4}{3} \mu \frac{dv}{dx} \right) \frac{dv}{dx}$$

For simplicity, the gas is here considered to be monoatomic. This does not lessen the generality of the analysis. For determination of the integration constants we have the following system of conditions

$$\rho v = D_1 \frac{dp}{dx} + D_2 \frac{dT}{dx}, \quad T = T_2, \quad T \frac{dp}{dx} + p \frac{dT}{dx} = 0 \quad \text{for } x = +l \quad (9.3)$$

$$\rho v = D_1 \frac{dp}{dx} + D_2 \frac{dT}{dx}, \quad T = T_1, \quad T \frac{dp}{dx} + p \frac{dT}{dx} = 0 \quad \text{for } x = -l \quad (9.4)$$

$$\frac{1}{2l} \int_{-l}^{+l} \rho dx = \rho_0 \quad (9.5)$$

At first glance it may seem that we have seven conditions for the determination of six integration constants.

However, examination of the first equation in (9.2) shows us at once that the first of the conditions in (9.3) and (9.4) are not independent but follow one from the other on the strength of this differential equation.

Since magnitude ϵ^* is assumed to be small, it is natural to linearize equations (9.2) and conditions (9.3) - (9.5). We put

$$\rho = \rho_0 + \rho', \quad T = \frac{1}{2}(T_1 + T_2) + T' = T_0 + T', \quad v = v' \quad (9.6)$$

$$D_i = D_i^{(0)} + D_i', \quad K_i = K_i^{(0)} + K_i', \quad \mu = \mu_0 + \mu'$$

where the quantities with prime marks are small.

Then we obtain the following system of equations for the determination of quantities ρ' , v' and T' :

$$\rho_0 \frac{dv'}{dx} = D_2^{(0)} \frac{d^2 \rho'}{dx^2} + D_2^{(0)} \frac{d^2 T'}{dx^2}$$

$$\frac{4}{3} \mu_0 \frac{d^2 v'}{dx^2} - R \left(T_0 \frac{d\rho'}{dx} + \rho_0 \frac{dT'}{dx} \right) = 0 \quad (9.7)$$

$$K_1^{(0)} \frac{d^2 \rho'}{dx^2} + K_2^{(0)} \frac{d^2 T'}{dx^2} = c_p \rho_0 T_0 \frac{dv'}{dx}$$

$$R = c_p - c_v \quad (9.8)$$

Moreover, we will have the following system of conditions for the determination of the integration constants:

$$\rho_0 v' = D_1^{(0)} \frac{d\rho'}{dx} + D_2^{(0)} \frac{dT'}{dx} \quad \text{at } x = +l$$

$$T' = \frac{T_2 - T_1}{2}, \quad T_0 \frac{d\rho'}{dx} + \rho_0 \frac{dT'}{dx} = 0 \quad \text{at } x = +l$$

$$T' = -\frac{T_2 - T_1}{2}, \quad T_0 \frac{d\rho'}{dx} + \rho_0 \frac{dT'}{dx} = 0 \quad \text{at } x = -l \quad (9.9)$$

$$\int_{-l}^{+l} \rho' dx = 0$$

Integrating the first and third of equations (9.7) and taking into consideration the first of conditions (9.9), we get

$$D_1^{(0)} \frac{d\rho'}{dx} + D_2^{(0)} \frac{dT'}{dx} = \rho_0 v' \quad (9.10)$$

$$K_1^{(0)} \frac{d\rho'}{dx} + K_2^{(0)} \frac{dT'}{dx} = c_p \rho_0 T_0 v' + A \mu_0 c_p T_0$$

where $A \mu_0 c_p T_0$ is the integration constant. Since by virtue of (6.2)

$$D_1^{(0)} = \frac{\mu_0}{\rho_0} \alpha_1, \quad K_1^{(0)} = \mu_0 c_p \beta_1 \frac{T_0}{\rho_0}$$

$$D_2^{(0)} = \frac{\mu_0}{T_0} \alpha_2, \quad K_2^{(0)} = \mu_0 c_p \beta_2 \quad (9.11)$$

solving (9.10) for quantities $\frac{dp'}{dx}$ and $\frac{dT'}{dx}$, we get

$$\begin{aligned} \frac{dp'}{dx} &= \frac{P_0}{\Delta} \left[\left(\beta_2 - \frac{c_p}{c_v} \alpha_2 \right) \frac{P_0}{M_0} v' - \alpha_2 A \right] \\ \frac{dT'}{dx} &= \frac{T_0}{\Delta} \left[- \left(\beta_1 - \frac{c_p}{c_v} \alpha_1 \right) \frac{P_0}{M_0} v' + \alpha_1 A \right] \end{aligned} \quad \left(\Delta = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \right) \quad (9.12)$$

Introducing (9.12) into the second of equations (9.7), we get

$$v^2 \frac{d^2 v'}{dx^2} - v' = -uA \quad (9.13)$$

In this equation

$$v^2 = \frac{A}{3} \frac{-\Delta}{(c_p/c_v)(\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)} \frac{M_0^2}{P_0 \rho_0} \quad (9.14)$$

$$u = \frac{\alpha_2 - \alpha_1}{(c_p/c_v)(\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)} \frac{M_0}{\rho_0} \quad (p_0 = R \rho_0 T_0)$$

Equation (9.13) is easily integrated. After integration we obtain

$$v' = uA + B_1 e^{+x/v} + B_2 e^{-x/v} \quad (9.15)$$

where B_1 and B_2 are integration constants.

With the aid of (9.15) and (9.12) it is easy to find and T' . Having performed the calculations, we obtain

$$\rho' = \frac{\rho_0 A}{(c_p/c_v)(\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)} x + \frac{\rho_0}{\Delta} \left(\beta_2 - \frac{c_p}{c_v} \alpha_2 \right) \nu \frac{\rho_0}{\mu_0} (B_1 e^{4\nu} - B_2 e^{-4\nu}) + C_1 \quad (9.16)$$

$$T' = \frac{-T_0 A}{(c_p/c_v)(\alpha_2 - \alpha_1) - (\beta_2 - \beta_1)} x - \frac{T_0}{\Delta} \left(\beta_1 - \frac{c_p}{c_v} \alpha_1 \right) \nu \frac{\rho_0}{\mu_0} (B_1 e^{4\nu} - B_2 e^{-4\nu}) + C_2$$

where C_1 and C_2 are integration constants.

Having determined the integration constants A , B_1 , B_2 , C_1 and C_2 , from the five remaining unutilized conditions we obtain for the determination of the integration constants

$$v' = \frac{T_2 - T_1}{2l} \frac{\mu_0}{\rho_0 T_0} (\alpha_2 - \alpha_1), \quad \rho' = -\frac{T_2 - T_1}{T_2 + T_1} \rho_0 \left(\frac{v}{l} \right), \quad T' = \frac{T_2 - T_1}{2} \left(\frac{v}{l} \right) \quad (9.17)$$

With these formulas it is now simple to find the flow t_x of heat through any region perpendicular to the x axis. Applying formula (5.7) in linearized form and using formula (6.2), we

shall obtain

$$t_x = M_0 c_v (\beta_2 - \beta_1) \frac{T_2 - T_1}{2l} \quad (9.18)$$

From formulas (9.17) and (9.18) one may draw certain conclusions.

In the first place, if we succeed in finding the magnitude of v' by experiment, then with the first formula in (9.17) we shall have a means of determining the difference $\alpha_2 - \alpha_1$.

In the second place, it follows directly from formula (9.18) that the measurement of the flow t_x of heat permits the determination of $\beta_2 - \beta_1$.

And, finally, in the third place, it follows from formula (6.23) that all experimental works devoted to the search for the magnitude of β_2 according to the formula

$$\beta_2 = \frac{2lt_x}{(T_2 - T_1)M_0 c_v} \quad (9.19)$$

which has only been employed in experimental works, have in fact given the magnitude of $\beta_2 - \beta_1$.

10. SECOND EXAMPLE OF INTEGRATION OF A SYSTEM OF GAS MOTION EQUATIONS

Let us now consider the one-dimensional problem of the steady transmission of heat through a layer of gas situated between two parallel planes, with the conditions that mass forces are absent, but the difference between the temperatures of the partitions is not small.

In accordance with the kinetic theory of gases we shall consider that the viscosity μ depends only on the temperature and is a known function of the temperature. In this case we must again integrate the system of equations (9.2) using conditions (9.3) - (9.5) in order to determine the arbitrary integration constants.

On the basis of the first of formulas (9.17), which also gives the order of velocity v of the gas for the case being considered, we may affirm that in an extremely broad class of cases the velocity will be a small magnitude even for a large temperature drop.

Making use of this, let us disregard the terms in the second and third of equations (9.2) which are of the order of μ^2 and μ^3 . Then the system of equations (9.2) is converted to the system of equations

$$\frac{d}{dx}(\rho v) = \frac{d}{dx} \left(D_1 \frac{dp}{dx} + D_2 \frac{dT}{dx} \right) \quad (10.1)$$

$$\frac{d}{dx} R\rho T = 0$$

$$c_v \frac{d}{dx} (p\rho T) = \frac{d}{dx} \left(K_1 \frac{dp}{dx} + K_2 \frac{dT}{dx} \right) - R\rho T \frac{dv}{dx} \quad (10.1) \text{ (con't)}$$

The first of equations (10.1) is integrated directly and after satisfaction of the first marginal condition in (9.3) and also, consequently, of the first marginal condition in (9.4), gives

$$p\rho = D_1 \frac{dp}{dx} + D_2 \frac{dT}{dx} \quad (10.2)$$

The second of equations (10.1) is also integrated and gives

$$R\rho T = C_1 \quad (10.3)$$

where C_1 is an integration constant.

The integral (10.3) provides for the fulfillment of the third condition in both (9.3) and (9.4)

Moreover, if (10.3) is taken into account, the third equation in (9.1) [sic -- should read (10.1)] is also integrated and gives the relationship

$$c_v p\rho T = K_1 \frac{dp}{dx} + K_2 \frac{dT}{dx} - C_1 v = C_2 \quad (10.4)$$

where C_2 is an integration constant.

Let us insert the velocity v from (10.2) into equation (10.4). Let us make use of the expressions (6.2) for coefficients D_1 , D_2 , K_2 and K_1 , insert them into (10.4) and, finally, eliminate the density ρ from (10.4) by the aid of (10.3). Then we obtain the extremely simple equation:

$$\mu \frac{dT}{dx} = \frac{C_2}{c_v(\beta_2 - \beta_1) - c_p(\alpha_2 - \alpha_1)} \quad (10.5)$$

Integrating this equation and satisfying the second marginal conditions in (9.3) and (9.4), we obtain

$$\left[\int_{T_2}^{T_1} \mu(T) dT \right]^{-1} \left[\int_{T_1}^T \mu(T) dt - \int_T^{T_2} \mu(T) dT \right] = \frac{x}{l} \quad (10.6)$$

The right hand side of (10.6) is a known function of temperature, and therefore from (10.6) it is possible to find the distribution of temperature in relation to x .

Using (10.3) and condition (9.5) and converting from integration with respect to a variable x to integration with respect to a variable T by the aid of (10.5), we find the value of constant C_1 and obtain the formula for density:

$$\rho = \frac{\rho_0}{T} \int_{T_2}^{T_1} \mu(T) dT \left[\int_{T_1}^{T_2} \frac{1}{T} \mu(T) dt \right]^{-1} \quad (10.7)$$

Further, introducing the expressions for T and ρ in (10.2), using (10.5) and the expression for C_2 , we find an expression for v . We get

$$v = \frac{\alpha_2 - \alpha_1}{2(\rho)} \int_{T_1}^{T_2} \frac{1}{T} \mu(T) dT \quad (10.8)$$

Formulas (10.6), (10.7) and (10.8) solve the problem that was posed.

Finally, we may also obtain without difficulty an expression for the flow of heat t_x . It, like the velocity, is found to be constant and is given by the formula

$$t_x = C_2 (\beta_2 - \beta_1) \frac{1}{2l} \int_{T_1}^{T_2} \mu(T) dT. \quad (10.9)$$

Let us make a few remarks about the formulas we have obtained.

First of all let us note that we too are unable to find the magnitudes of α_1 , α_2 , β_1 and β_2 separately and can only find their differences from these formulas. Let us note further that formulas (10.6) and (10.7) permit us to find easily the dependence of viscosity upon temperature by means of measurements of density or temperature.

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$$Q_{np} = f_1 \lambda c \frac{\partial \rho}{\partial n} \quad (2.5)$$

where f_1 is a nondimensional factor of the order of unity.

From formula (2.5) we see that with variation of density in a macroscopically quiescent gas there occurs a flow of mass through an area which is stationary relative to the gas. This phenomenon of flow of mass as a result of density variation may for expediency be called density self-diffusion.

It is evident that the reasoning also holds if we consider a macroscopically moving gas and an area moving in space with the macroscopic velocity \mathbf{v} of the gas. The density self-diffusion flow of mass, Q_{np} , will be given by formula (2.5) in this case also.

Let us remark that density self-diffusion has never been taken into account in the writing of equations of motion of a viscous gas.

2. Density Heat Transfer

Let us consider, as above, a certain stationary region of area ΔS with normal \mathbf{n} in a macroscopically quiescent gas, retain the former designations and analyze the problem of transfer of internal energy through area ΔS , assuming the temperature