Problem Concerning the Fundamental Equations of the Relativistic Quantum Theory of the Field

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In present works on the quantum theory, fields proceed from the representation for the interaction in which the wave vector Φ is considered as a functional of the spatially similar surface σ and Schrödinger's equation assumes the form:

$$i\hbar\frac{\partial\Phi(\sigma)}{\partial\sigma_x} = H(x,\sigma)\Phi(\sigma).$$

It is very clear, however, from considerations of covariancy that the introduction of arbitrary spatially similar surfaces is in general superfluous and that it is limited completely sufficiently by the class of spatially similar hyperplanes.

Such hyperplanes can be given by the equation

$$x\xi = x_0\xi_0 - \sum_{(1\leqslant\alpha\leqslant 3)} x_\alpha \xi_\alpha = \tau$$

with unit temporally similar vector $\{\xi\}$, by characterizing them by the scalar τ and by the three spatial components $\{\xi_\alpha\}$. In this way the wave vector becomes a function of the four variables indicated.

If we desire to preserve the most characteristic features of the present theory — the strict determinization of the evolution of the wave vector and the preservation of its norm during transition from one hyperplane to another — then we must write the fundamental wave equations, for example, in the following form:

$$i\hbar\frac{\partial\Phi(\tau,\xi)}{\partial\tau} = H(\tau,\xi)\Phi(\tau,\xi),$$
$$i\hbar\frac{\partial\Phi(\tau,\xi)}{\partial\xi^\alpha} = H_\alpha(\tau,\xi)\Phi(\tau,\xi) \qquad \alpha = 1,2,3, \tag{1}$$

in which the operators H, H_α must be Hermitian. Here we can consider, as is usual, that H, H_α are expressions depending upon the operators of generation and annihilation of free particles. The conditions governing the compatibility of the system of equations (1) will be:

$$i\hbar\frac{\partial H}{\partial\xi^\alpha} + HH_\alpha = i\hbar\frac{\partial H_\alpha}{\partial\tau} + H_\alpha H$$
$$i\hbar\frac{\partial H_\alpha}{\partial\xi^\beta} + H_\alpha H_\beta = i\hbar\frac{\partial H}{\partial\xi^\alpha} + H_\beta H_\alpha \; . \tag{2}$$

In order to formulate the requirement of the relativistic covariancy of these equations, we introduce the unitary operator U_L with whose aid we transform the operators of the free particles during transformation of spaces by the Lorentz transformation $L = L_{rot}L_{tr}$ $(L_{tr}x = x+a)$. This condition then can be written down in the following form:

$$\overset{+}{U}_L H(\tau,\xi)U_L = H(\tau+a\xi, L_r\xi),$$
$$\sum_{(1\leqslant\alpha\leqslant 3)} \overset{+}{U}_L H(\tau,\xi)U_L \delta\xi_\alpha = \sum_{(1\leqslant\alpha\leqslant 3)} H_\alpha(\tau+a\xi, L_r\xi)(\delta L_r\xi)_\alpha \; , \tag{3}$$

hence it follows that the transformation wave vector

$$\Phi'(\tau,\xi) = U_L \Phi(\tau + a\xi,\ L_r\xi) \tag{4}$$

also satisfies equations (1).

Thus with the aid of (4) we can determine the transformation of the wave vector for the Lorentz transformation $x \to Lx$.

On the basis of (4) it is also easy to obtain the expression for the components of the total impulse (momentum) and moment of the system.

For this purpose it is only necessary to consider an infinitesimally small transformation (Lorentz):

$$\begin{gathered} x_0 \to x_0 + \delta a + \sum_{(\alpha<\beta<3)} g_\alpha x_\beta \delta w_{\alpha\beta} \quad (\alpha = 0,1,2,3) \\ g_0 = 1, \quad g_1 = g_2 = g_3 = -1, \quad \delta w_{\alpha\beta} + \delta w_{\beta\alpha} = 0 \end{gathered} \tag{5}$$

and the corresponding infinitely small transformation

$$U_L = 1 + \frac{1}{i\hbar}\left((P^0 \cdot \delta a) - \frac{1}{2}\sum_{\alpha,\beta} M^0_{\alpha\beta}\delta w_{\alpha\beta}\right),$$

where P^0 is the total impulse (momentum) and $M^0_{\alpha\beta}$ is the components of the moment for the system of non-interacting particles.

As follows from (4), for an infinitely small transformation (5) the wave vector suffers the increment:

$$\delta\Phi(\tau,\xi) = \frac{1}{i\hbar}\left((P^0 \cdot \delta a) - \frac{1}{2}\sum_{\alpha,\beta} M^0_{\alpha\beta}\delta w_{\alpha\beta} + (\xi \cdot \delta a)H - \sum_{\substack{(1\leq\alpha\leq 3)\\(0\leq\beta\leq 3)}} H_\alpha \xi_\beta \delta w_{\alpha\beta}\right)\Phi(\xi,\tau).$$

Hence for the components of the total impulse (momentum) and moment we obtain, taking into consideration the interaction, the following expressions:

$$\begin{gathered} P_\alpha = P^0_\alpha + \xi_\alpha H(\tau,\xi) \\ M_{\alpha\beta} = M^0_{\alpha\beta} + (\xi_\alpha H_\beta(\tau,\xi) - \xi_\alpha H_\beta(\tau,\xi)) \quad (\alpha,\beta = 1,2,3) \\ M_{\alpha 0} = M^0_{\alpha 0} + \xi_0 H_\alpha(\tau,\xi) \qquad (\alpha = 1,2,3). \end{gathered} \tag{6}$$

It is easy to verify that the expressions obtained actually possess the necessary properties for representing the components of impulse (momentum) and moment. For example, their averages taken over the wave vector $\Phi(\tau,\xi)$ does not depend upon ξ; the average impulse (momentum) is transformed as a four-vector etc.

Let us stress here that the main problem now is the actual construction of equations (1) with the operators H, H_α satisfying the requirements mentioned earlier.

In order to arrive at its solution we assume for the moment that we can construct the unitary operator $S(\tau,\xi)$ satisfying the following condition:

$$U_L^{+} S(\tau,\xi) U_L = S(\tau + a\xi, L_r\xi).$$

Then it is clear that the expressions $H = i\hbar\frac{\partial S}{\partial\tau}S^{+}, \quad H_0 = i\hbar\frac{\partial S}{\partial\xi_\alpha}S^{+}$

satisfy all the requirements expounded.

This remark, however, is vitiated by the fact that in all physically interesting cases such an operator as $S(\tau,\xi)$ does not exist.

More accurately, if such an operator does exist, then all energy levels of the considered system will with the presence of interaction be the same as during the complete absence of interaction.

We can attempt to bypass the indicated difficulty by noting that we do not need at all that there should exist an operator itself $S(\tau,\xi)$ which reduces the wave vector of free non-interacting particles to $\Phi(\tau,\xi)$; it is only necessary that the symbolic product $S(\tau,\xi)\overset{+}{S}(\tau',\xi')$ should possess meaning, which represents the operator of the transformation of the wave vector with the hyperplane (τ',ξ') to the hyperplane (τ,ξ).

Let us consider the formal expansion:

$$S(\tau,\xi) = 1 + \frac{1}{i\hbar}S(\tau,\xi) + \cdots + \left(\frac{1}{i\hbar}\right)^n S_n(\tau,\xi) + \cdots \tag{7}$$

such that

$$\overset{+}{U}_L S_n(\tau,\xi) U_L = S_n(\tau + a\xi, L_r\xi) \tag{8}$$

and

$$\overset{+}{S}_1 = S_1, \quad S_n + (-)^n \overset{+}{S}_n + \sum_{(1<k<n-1)} (-)^{n-k} S_k \overset{+}{S}_{n-k} = 0. \tag{9}$$

Then the expressions following satisfy formally our conditions

$$\begin{aligned} H &= \frac{\partial S_1}{\partial \tau} + \frac{1}{i\hbar}\left(\frac{\partial S_2}{\partial \tau} - \frac{\partial S_1}{\partial \tau}\overset{+}{S}_1\right) + \cdots + \left(\frac{1}{i\hbar}\right)^n \sum_{(1<k<n)} \frac{\partial S_k}{\partial \tau}\overset{+}{S}_{n-k}(-1)^{n-k} + \cdots \\ H_o &= \frac{\partial S_1}{\partial \xi^\alpha} + \frac{1}{i\hbar}\left(\frac{\partial S_2}{\partial \xi^\alpha} - \frac{\partial S_1}{\partial \xi^\alpha}\overset{+}{S}_1\right) + \cdots + \left(\frac{1}{i\hbar}\right)^n \sum_{(1<k<n)} \frac{\partial S_k}{\partial \xi^\alpha}\overset{+}{S}_{n-k}(-1)^{n-k} + \cdots \end{aligned} \tag{10}$$

The expressions for H, H are thus obtained in the form of series; even ordinary equations, however, of the quantum theory of fields contain expansions of a number of quantities, for example field mass and charge, which are employed for renormalization.

Now arises the problem of selecting $S(\tau,\xi)$ such that the series (10) should possess meaning and the following series should converge

$$S(\tau',\xi')\overset{+}{S}(\tau,\xi) = 1 + \cdots + \left(\frac{1}{i\hbar}\right)^n \sum_{(0<k<n)} (-1)^{n-k} S_k(\tau',\xi')\overset{+}{S}_{n-k}(\tau,\xi) + \cdots \tag{11}$$

if only for τ',ξ' sufficiently close to τ,ξ. This latter condition would ensure the possibility of the integration of the fundamental equations (1).

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