"Application of A. M. Lyapunov's Theory of Stability to the Theory of Differential Equations with Small Multipliers in the Derivatives

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APPLICATION OF A. M. LYAPUNOV'S THEORY STABILITY TO THE THEORY OF DIFFERENTIAL EQUATIONS WITH SMALL MULTIPLIERS IN THE DERIVATIVES

I. S. Gradshteyn

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In my article (2) I indicated the connection of the theory of stability of differential equations with small multipliers (factors) in the derivatives with the first method of A. M. Lyapunov /Liapounoff/ in his investigation of the stability of motion. Further investigations have shown that this connection concerns not only the first method but the entire theory of A. M. Lyapunov (1).

1. Definitions. Let $(\vec{y}^*, \vec{p}^*)$ represent a set of singular points of the family of systems of equations

$$dY_i/dt = h_i(\vec{Y},\vec{p}) \quad (i = 1,2, \ldots,n) \tag{1}$$

where $\vec{Y}$ is a n-dimensional vector and $\vec{p}$ is a ℓ-dimensional vector-parameter.

For the points of the set R representing the part (properly or improperly) of set $(\vec{y}^*, \vec{p}^*)$, let the solution of the system (1) possess the following properties: for any $e > 0$ and $0 < a \leq e$ one can show $0 < r(e) \leq e$ and $\Theta(r,a) > 0$ such that for any system of family (1), the singular point

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of which enters R, from the inequalities

$$/Y_i(0,\vec{p}) - y_i^* / < r \quad (i = 1,2,\ldots,n) \tag{2}$$

for $t \geqslant 0$, follow the inequalities

$$/Y_i(t,\vec{p}) - y_i^* / < e \quad (i = 1,2,\ldots,n) \tag{3}$$

but for $t \geqslant \Theta$, follow the inequalities

$$/Y_i(t,p) - y_i^* / < a \quad (i = 1,2,\ldots n) . \tag{4}$$

If these conditions are fulfilled we shall say that the family of equations (1) defines a motion asymptotically stable relative to the set R.

Here we naturally assume that in a certain neighborhood of the set R the functions h_i satisfy the Lipschitz conditions relative to $\vec{y}$.

2. Theorem. The theorem which I published in (3), Section 2, remains true if:

a) we define the set R with the following property: the motion determined by the family of the systems of differential equations

$$dy_i/dt = h_i(\vec{X},\vec{y}, T), \quad (i = 1,2,\ldots,n) \tag{5}$$

in which $\vec{x}$ and t appear as parameters, is uniformly asymptotically stable relative to the set R;

b) we assume that $D(h_1,h_2,\ldots,h_n) / D(y,y_2,\ldots,y_n) \neq 0$ in the set R.

3. Criteria governing uniform asymptotic stability are obtained naturally from the criteria governing A. M. Lyapunov's stability. The criterion based on his first method was actually published by the author in previous works (2,3). The criterion based on the second method is the following:

a) the function $V(\vec{q}; \vec{x}^*, \vec{y}^*, T^*)$ is defined for $/q_i/ \leqslant H$ (where H is an arbitrarily small fixed positive number) and for values of the parameters $\vec{x}^*$, $\vec{y}^*, T^*$ determining the points of a certain compactum $R^* \subset R$;

b) the function $V(\vec{q}; \vec{x}^*, \vec{y}, T^*)$ in the region of its definition is continuous in all variables and possesses derivatives of the first order in $\vec{q}$ continuous in all variables;

c) the function V and $-\sum_{i=1}^{n} \frac{\partial V}{\partial q_i} \cdot h_i(\vec{x}^*, \vec{y}^* + \vec{q}, T^*)$ are positive definite functions.

In such a case the family of systems of equations (1) or, what is the same, $\vec{y} = \vec{y}^* + \vec{q}$,

$$dq_i/dt = h_i(\vec{x}^*, \vec{y}^* + \vec{q}, T^*),$$

defines motions that are uniformly asymptotically stable relative to the compactum R.

The theorem of A. N. Tikhonov (4) on the existence is a partial case of this theorem; corresponding to Lyapunov's function

$$V = \sum_{i=1}^{n} q_i^2 .$$

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