member depends on the force of gravity g, the length of the wave λ and the ratio h/λ (the depth of the layer h to the length of the wave λ). This term is called the "dynamic term";

$$c = \pm \sqrt{\frac{g\lambda}{2\pi}\, th\, \frac{2\pi h}{\lambda}}. \tag{23}$$

The algebraic signs ahead of the radical indicate that the wave can be propagated in a positive as well as a negative direction.

Let us consider two ultimate cases, namely, when the waves are propagated on the surface of a very deep liquid and on the surface of a very shallow liquid.

For very great values of x

$$th\, x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \approx 1,$$

and for very small values of x

$$th\, x = \frac{1 + x - (1 - x)}{(1 + x) + (1 - x)} = x.$$

Consequently for a very deep liquid

$$c = \sqrt{\frac{g\lambda}{2\pi}}, \tag{24}$$

and for a very shallow liquid

$$c = \sqrt{gh}, \tag{25}$$

Thus, we derived the Eri formula (24) for the velocity of wave propagation on the surface of very deep water, and the Lagrange

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formula (25) for the propagation velocity of "long waves" on the surface of shallow water.

Table 53 furnishes the propagation velocities for waves of various lengths at a varied depth of the liquid layer, the computation made in accordance with formula (23)

/See next page for Table 53/

Let us analyze in more detail as to what is the field of application of formulas (24) and (25).

When $h/\lambda = 0.42$, the value of $th(2\pi h/\lambda) = 0.99$, i.e., only by one percent less than $2\pi h/\lambda$, then, velocity c obtained as per formula (24) will be only by 0.5 percent greater than the propagation velocity of the wave, as determined by formula (23). Consequently, formula (24) is of adequate precision, if the depth of the liquid is no less that o.4 of the length of the wave ($h > 0.4\lambda$). This shows what is meant by the concept of a very deep liquid, which concept without the above evaluation might have been misleading. If $h/\lambda = 0.027$, the values $th(2\pi h/\lambda)$ will differ from $2\pi h/\lambda$ only by one percent. Therefore, velocity c, computed by the approximation formula (25), will differ from the precise value of c, as determined by formula (23), by less than 0.5 percent.

With relation to the atmosphere, we will frequently use the Lagrange formula (25) for the approximate determination of the propagation velocity of a wave, since, due to the vast horizontal range of the atmosphere as compared to the thickness of atmospheric layers, the ratic $h/\lambda \ll 1$.

TABLE 53

Propagation Velocity of Gravitational Waves on a Surface of a Liquid at Rest

h \ λ	1	10	100	500	1000	5000	10,000	100,000	100,000	∞
1	1	3	3	3	3	3	3	3	3	3
10	1	4	9	10	10	10	10	10	10	10
100	1	4	12	26	29	31	31	31	31	31
1000	1	4	12	28	39	81	93	99	99	99
10,000	1	4	12	28	39	88	125	295	313	313
∞	1	4	12	28	39	88	129	394	1250	

As an example, let us compute the propagation velocity of a wave on the surface of a homogeneous atmosphere. The height of the homogeneous atmosphere is $H = RT_0/g$. Therefore, for $T_0 = 293$

$c = \sqrt{RT_0}$,

$$c = \sqrt{RT_0} = \sqrt{293 \cdot 287} = 290 \text{ meters/second.}$$

Thus, the propagation velocity of long waves on the free surface of a homogeneous atmosphere is equal to the velocity of sound.

Section 5. Flat Waves on a Surface of Separation of Two Currents.

We will now analyze a wave motion, developing on a surface dividing two liquids, which are of different densities ρ_1, ρ_2 and have different velocities U_1 and U_2 of their basic motion.

In the atmosphere, such conditions correspond to waves developing on a surface of discontinuity.

In order that a wave motion develops on a surface of discontinuity it is necessary that below the surface be a colder air mass, and above the surface - a warmer air mass. Designating, as in the previous section, by index 1, the magnitudes relating to the warm air mass, and by index 2, the magnitudes relating to the cold air mass, we will assume that the first air mass is above, and the cold air mass is below. For the time being, we will disregard the earth's rotation, so that the surface of discontinuity in a non-perturbed state will be horizontal.

Let the surface of discontinuity be the plane Oxy, we place the origin of coordinates on this plane, we direct the positive z-axis vertically up, and the x-axis in the direction of the motion of both liquids.

Assuming that the upper liquid is restricted from the top by a rigid boundary - plane, the equation of which is $z = h_1$, and that the lower liquid is also restricted by a rigid plane, the equation of which is $z = -h_2$.

Assuming the motion to be flat, the velocity components of the motion under study will be, for the upper liquid, $U_1 + u_1$, w_1, and for the lower liquid $U_2 + u_2$, w_2, the pressure in the upper liquid $P_1 + p_1$, the pressure in the lower liquid $P_2 + p_2$. Each one of the liquids is assumed to be non-compressible, so that $\rho_1 =$ const., $\rho_2 =$ const., with $\rho_1 > \rho_2$. The equation of the surface of separation with the small perturbations superposed on it will be $z = \zeta(x, t)$.

Now then, the equations of small perturbations will assume the form

$$\left.\begin{aligned}
&\frac{\partial u_1}{\partial t} + U_1 \frac{\partial u_1}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}; \quad \frac{\partial u_2}{\partial t} + U_2 \frac{\partial u_2}{\partial x} = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial x}, \\
&\frac{\partial w_1}{\partial t} + U_1 \frac{\partial w_1}{\partial x} = -\frac{1}{\rho_1}\frac{\partial p_1}{\partial z}; \quad \frac{\partial w_2}{\partial t} + U_2 \frac{\partial w_2}{\partial x} = -\frac{1}{\rho_2}\frac{\partial p_2}{\partial z}, \\
&\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \quad \frac{\partial u_2}{\partial x} + \frac{\partial w_2}{\partial z} = 0.
\end{aligned}\right\} \quad (1)$$

Boundary conditions will be:

$$\left.\begin{aligned}
w_1 &= 0, \quad \text{when } z = h_1, \\
w_2 &= 0, \quad \text{when } z = h_2
\end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned} p_1 - p_2 &= g(\rho_1 - \rho_2)\zeta, \\ w_1 &= \frac{\partial \zeta}{\partial t} + U_1 \frac{\partial \zeta}{\partial x}, \\ w_2 &= \frac{\partial \zeta}{\partial t} + U_2 \frac{\partial \zeta}{\partial x} \end{aligned}\right\} \text{when } z = 0 \qquad (3)$$

In the same way as in the case of one layer of liquid, we will seek the solution in the form

$$\left.\begin{aligned} u_1(x,z,t) &= \bar{u}_1(z)e^{i(\sigma t - \kappa x)}, \quad u_2(x,z,t) = \bar{u}_2(z)e^{i(\sigma t - \kappa x)}, \\ w_1(x,z,t) &= \bar{w}_1(z)e^{i(\sigma t - \kappa x)}, \quad w_2(x,z,t) = \bar{w}_2(z)e^{i(\sigma t - \kappa x)}, \\ p_1(x,z,t) &= \bar{p}_1(z)e^{i(\sigma t - \kappa x)}, \quad p_2(x,z,t) = \bar{p}_2(z)e^{i(\sigma t - \kappa x)}, \\ \zeta(x,t) &= \zeta_0 e^{i(\sigma t - \kappa x)}. \end{aligned}\right\} \qquad (4)$$

Repeating the argumentations of the preceding sections, and satisfying condition (2) at the rigid boundaries, and considering only the substantial part we derive:

$$\left.\begin{aligned} u_1 &= C_1 \operatorname{ch} \kappa(z+h_1)\cos(\sigma t - \kappa x); \\ u_2 &= C_2 \operatorname{ch} \kappa(z-h_2)\cos(\sigma t - \kappa x); \\ w_1 &= -\frac{C_1 \kappa}{\rho_1(\sigma - \kappa U_1)} \operatorname{sh} \kappa(z+h_1)\sin(\sigma t - \kappa x); \\ w_2 &= -\frac{C_2 \kappa}{\rho_2(\sigma - \kappa U_2)} \operatorname{sh} \kappa(z-h_2)\sin(\sigma t - \kappa x); \\ p_1 &= C_1 \operatorname{ch} \kappa(z+h_1)\cos(\sigma t - \kappa x); \\ p_2 &= C_2 \operatorname{ch} \kappa(z-h_2)\cos(\sigma t - \kappa x); \\ \zeta &= \zeta_0 \cos(\sigma t - \kappa x). \end{aligned}\right\} \qquad (5)$$

Satisfying the boundary conditions (3), we derive:

$$\left.\begin{aligned} &C_1\, ch\, \varkappa h_1 - C_2\, ch\, \varkappa h_2 = g(\rho_1 - \rho_2)\zeta_0, \\ &-\frac{C_1 \varkappa}{\rho_1(\sigma - \varkappa U_1)}\, sh\, \varkappa h_1 = \zeta_0(\sigma - \varkappa U_1), \\ &\frac{C_2 \varkappa}{\rho_2(\sigma - \varkappa U_2)}\, sh\, \varkappa h_2 = \zeta_0(\sigma - \varkappa U_2). \end{aligned}\right\} \qquad (6)$$

From the last two equations (6) we determine the arbitrary constants C_1 and C_2

$$\left.\begin{aligned} C_1 &= -\zeta_0 \frac{\rho_1}{\varkappa\, sh\, \varkappa h_1}(\sigma - \varkappa U_1)^2, \\ C_2 &= \zeta_0 \frac{\rho_2}{\varkappa\, sh\, \varkappa h_2}(\sigma - \varkappa U_2)^2. \end{aligned}\right\} \qquad (7)$$

Substituting the values for C_1 and C_2 into the first equation (6), we derive an equation which determines the wave frequency:

$$\rho_1(\sigma - \varkappa U_1)^2 cth\, \varkappa h_1 + \rho_2(\sigma - \varkappa U_2)^2 cth\, \varkappa h_2 = g\varkappa(\rho_2 - \rho_1). \qquad (8)$$

Assuming for the sake of brevity that

$$cth\, \varkappa h_1 = cth \frac{2\pi h_1}{\lambda} = a_1,$$

$$cth\, \varkappa h_2 = cth \frac{2\pi h_2}{\lambda} = a_2,$$

we formulate equation (8) like this:

$$\sigma^2(\rho_1 a_1 + \rho_2 a_2) - 2\sigma\varkappa(\rho_1 U_1 a_1 + \rho_2 U_2 a_2) + \\ + \varkappa^2\left\{\rho_1 U_1^2 a_1 + \rho_2 U_2^2 a_2 - \frac{g}{\varkappa}(\rho_2 - \rho_1)\right\} = 0. \qquad (9)$$

Solving the quadratic equation (9), we derive

$$\sigma = \varkappa \frac{\rho_1 U_1 a_1 + \rho_2 U_2 a_2}{\rho_1 a_1 + \rho_2 a_2} \pm \\ \pm \varkappa \sqrt{\left(\frac{\rho_1 U_1 a_1 + \rho_2 U_2 a_2}{\rho_1 a_1 + \rho_2 a_2}\right)^2 - \frac{\rho_1 U_1^2 a_1 + \rho_2 U_2^2 a_2}{\rho_1 a_1 + \rho_2 a_2} + \frac{g}{\varkappa}\frac{\rho_2 - \rho_1}{\rho_1 a_1 + \rho_2 a_2}}. \qquad (10)$$

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Or, dividing both parts of the equality by k, after obvious transpositions, we derive a formula, determining the propagation velocity of the waves c:

$$c = \frac{\rho_1 U_1 a_1 + \rho_2 U_2 a_2}{\rho_1 a_1 + \rho_2 a_2} \pm \pm \sqrt{\frac{g\lambda}{2\pi}\frac{\rho_2 - \rho_1}{\rho_1 a_1 + \rho_2 a_2} - \rho_1\rho_2 a_1 a_2 \left(\frac{U_1 - U_2}{\rho_1 a_1 + \rho_2 a_2}\right)^2}. \tag{11}$$

Formula (22) of the preceding section is a particular case of formula (11). Indeed, assuming $\rho_1 = 0$, and omitting indexes 2, we derive

$$c = U \pm \sqrt{\frac{g\lambda}{2\pi}\, th\, \frac{2\pi h}{\lambda}}.$$

By analogy with (22) of Section 4, the first item in formula (11) we will call the convective velocity of wave propagation. Obviously, the convective velocity has some intermediate value: it is greater than the minimum velocity of the currents and smaller than their maximum velocity. When both layers have great thickness,

$$a = cth\ kh = \frac{e^{kh} + e^{-kh}}{e^{kh} - e^{-kh}} = 1.$$

Then, the convective velocity c of the propagation of the wave will equal the mean velocity of the current.

$$c' = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \tag{12}$$

The second term of formula (11), in the case of layers of great thickenss, can be written over as:

$$c'' = \pm\sqrt{\frac{g\lambda}{2\pi}\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} - \rho_1\rho_2\left(\frac{U_1 - U_2}{\rho_1 + \rho_2}\right)^2}. \tag{13}$$

This velocity, by analogy with (23) of the preceding section, we will call the dynamic wave propagation velocity. As can be seen from (13), the dynamic velocity is stipulated by 2 factors: the discontinuity in density and the discontinuity in velocity. To ascertain the part played by individual factors, we shall analyze some individual cases,

Assuming that oscillations on a surface of separation occur about a state of rest, so that $u_1 = u_2 = 0$. Then:

$$c = \pm \sqrt{\frac{g\lambda}{2\pi} \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}}. \qquad (14)$$

This formula differs from the propagation velocity of the wave along a free surface by the presence of factor

$$\sqrt{\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}}$$

If we contemplate the waves on the surface of separation between water and air, it may be assumed that $\rho_2 = 1; \rho_1 = 0.0013$. Then:

$$\sqrt{\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}} = \sqrt{\frac{(\rho_2 - \rho_1)(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)(\rho_2 - \rho_1)}} \approx \frac{\rho_2 - \rho_1}{\rho_2} = 1 - 0.0013.$$

Therefore

$$c = \sqrt{\frac{g\lambda}{2\pi}}\,(1 - 0.0013).$$

Consequently, disregarding the wave motions on the part of the air, in determining c, we are committing an error, not exceeding 0.13 percent.

However, in the case of wave motions on a surface of separation in the atmosphere, entirely different conditions prevail.

In order to evaluate the order of magnitude of this factor, it becomes convenient to convert from density ρ to temperature with the aid of the Klapeyron equation $\rho_1 = p_1/RT_1,\ \rho_2 = p_2/RT_2$. Taking note of the fact that, on the surface of separation, pressure does not undergo a discontinuity $p_1 = p_2$; then we have

$$c = \pm\sqrt{\frac{g\lambda}{2\pi}\,\frac{T_1 - T_2}{T_1 + T_2}} \qquad (15)$$

When $T_1 = 283$, $T_2 = 273$, then

$$\sqrt{\frac{T_1 - T_2}{T_1 + T_2}} = \sqrt{\frac{10}{556}} = \sqrt{0.0180} \approx 0.13.$$

Thus, under these conditions

$$c = 0.13\sqrt{\frac{g\lambda}{2\pi}}.$$

Now then, on surfaces of separation occurring in the atmosphere, waves of the same length are propagated much more slowly than on a free surface.

It is necessary to take note of one important circumstance:

If $T_1 < T_2$, i.e., if the upper layer is colder than the lower one, the propagation velocity of the wave will be imaginary. This means that frequency σ will be imaginary.

Therefore, assuming $\sigma = is$, where s is a substantial magnitude, we derive for ζ :

$$\zeta = Ae^{st}\cos kx$$

Consequently, the perturbation will increase indefinitely with time. Therefore, the basic state is unstable and, consequently, a wave motion is impossible under conditions of non-stable stratification.

If the depth of both layers of the liquid is small as compared with the length of the wave, then

$$a_1 = cth\ \kappa h_1 = \frac{1}{\kappa h_1}, \qquad a_2 = cth\ \kappa h_2 = \frac{1}{\kappa h_2},$$

and, consequently, the formula for the propagation velocity of a wave on a surface of separation of two liquids at rest, will assume the form of

$$c = \sqrt{\frac{g}{\kappa}\,\frac{\rho_2 - \rho_1}{\frac{\rho_1}{\kappa h_1} + \frac{\rho_2}{\kappa h_2}}} = \sqrt{\frac{g(\rho_2 - \rho_1)\cdot h_1 h_2}{h_2 \rho_1 + h_1 \rho_2}}$$

or, by conversion from density to temperature:

$$c = \sqrt{\frac{g(T_1 - T_2) h_1 h_2}{h_1 T_1 + h_2 T_2}} = \sqrt{g h_2} \cdot \sqrt{\frac{T_1 - T_2}{T_1 + \frac{h_2}{h_1} T_2}} \,.$$

As also in the case of the infinitely great thickness of the liquid layers, the presence of the second layer leads to a decrease in the propagation velocity of the wave; so that when $h_1 = h_2$ for the same values of T_1 and T_2 we have:

$$c = 0.13 \sqrt{g h} \,.$$

Finally, we assume that the lower layer has a small thickness as compared with the length of the wave, and the upper layer extends upward indefinitely. Then

$$a_2 = \frac{1}{\kappa h_2}; \quad a_1 = 1,$$

and consequently:

$$c = \sqrt{\frac{g}{\kappa} \frac{\rho_2 - \rho_1}{\frac{\rho_2}{\kappa h_2} + \rho_1}} = \sqrt{g h_2 \frac{\rho_2 - \rho_1}{\rho_2}} = \sqrt{g h_2 \frac{(T_1 - T_2)}{T_1}} .$$

For the same values of T, and T_2 we derive:

$$c = 0.19 \sqrt{gh} .$$

Assuming now that there is no discontinuity in density, so that $\rho_1 = \rho_2 = \rho$, while there is a discontinuity in velocity $U_1 \neq U_2 \neq 0$. Then, for infinitely thick layers we derive from (11):

$$c = U_1 + U_2 \pm i(U_1 - U_2) . \tag{16}$$

Since the dynamic term now depends only on the discontinuity of velocity, such waves could be called "displacement waves". These waves can also be called inertia waves, since the wave motion in this case could only develop at the expense of the kinetic energy of the basic motion. However, the displacement waves, or the inertia waves, as can be seen from (16), are unstable.

Let us now turn to the more general case, when, on the surface of separation, there is discontinuity of wind as well as of density. Then, the dynamic propagation velocity will be determined as follows:

$$C'' = \sqrt{\frac{g\lambda}{2\pi}\frac{\rho_2-\rho_1}{\rho_2\alpha_2+\rho_1\alpha_1} - \rho_1\rho_2\alpha_1\alpha_2\left(\frac{U_1-U_2}{\rho_2\alpha_2+\rho_1\alpha_1}\right)^2}. \quad (17)$$

If the subradical expression is positive, the basic motion is stable; if the subradical expression is negative, the basic motion is unstable. Now then, the condition of non-stability of two currents, and, consequently, of the surface separating them, can be written down as:

$$\frac{g\lambda}{2\pi}\frac{\rho_2-\rho_1}{\rho_2\alpha_2+\rho_1\alpha_1} - \rho_1\rho_2\alpha_1\alpha_2\left(\frac{U_1-U_2}{\rho_2\alpha_2+\rho_1\alpha_1}\right)^2 < 0. \quad (18)$$

Let the length of the wave and the densities of both liquids be given: then, it is more convenient to solve the condition of non-stability about the discontinuity of velocity:

$$(U_1-U_2)^2 > \frac{g\lambda}{2\pi}(\rho_2-\rho_1)\left(\frac{1}{\rho_2\alpha_2}+\frac{1}{\rho_1\alpha_1}\right). \quad (19)$$

From this the conclusion can be drawn that, given the length of the wave and the discontinuity of density, the basic motion is unstable with a strong discontinuity of the velocity, and stable with a mild discontinuity of the velocity of the basic motion.

We shall now solve the condition of non-stability (19) with relation to λ, assuming the discontinuity of velocity as given. Then, the basic motion will be unstable, if the following inequality is fulfilled:

$$\lambda < \frac{2\pi}{g(\rho_2 - \rho_1)} \frac{(U_1 - U_2)^2 \rho_1 \rho_2 \alpha_1 \alpha_2}{\rho_1 \alpha_1 + \rho_2 \alpha_2}. \tag{20}$$

Now then, with a given discontinuity of velocity and a given discontinuing of density, the basic motion is unstable with relation to short waves, and stable with relation to long waves.

Let us also see what part with relation to the stability of the motion is played by the depth of the liquid layers. For this we interpolate into the condition of non-stability (20) the depth of the layers h_1 and h_2 in their explicit form; then this condition will change as follows:

$$\lambda < \frac{2\pi}{g(\rho_2 - \rho_1)} \frac{(U_1 - U_2)^2 \rho_1 \rho_2}{\rho_1 \, th \, \kappa h_2 + \rho_2 \, th \, \kappa h_1}. \tag{21}$$

Consequently, the basic motion is stable with relation to deep layers of liquid, and unstable with relation to shallow layers.

We note that for waves lying at the boundary of stability, the dynamic propagation velocity of the wave becomes zero.

Thus, the convective propagation velocity of the wave is lying at the stability boundary.

Let us write the expression for the boundary value of wave length λ, assuming both layers as infinitely deep, and interpolating, in place of density ρ, temperature T. Then we derive:

$$\lambda = \frac{2\pi}{g} \frac{(u_1 - u_2)^2 T_1 T_2}{T_1^2 - T_2^2}$$

Table 54 contains the ultimate values for the length of the wave under condition of instability, with $T = 273°$. The shorter waves are unstable, the longer ones are stable.

Table 54. Ultimate values for the length of wave in meters

Δu m/sec / ΔT °C	4	8	12	16	20
4	352	1408	3668	5616	8789
8	177	709	1595	2829	4426
12	119	475	1073	1901	2974
16	90	359	808	1431	2244
20	73	289	652	1156	1809

Gravitational waves are always stable with relation to the basic motion, while displacement waves are always unstable. Therefore, the gravitational force has a stabilizing effect upon the basic motion, and displacement has a labilizing effect. When the stabilizing effect of the gravitational force predominates over the labilizing effect of displacement, the motion is stable; in the opposite case, it becomes unstable.

Section 6. Small Oscillations of Surfaces of Separation.

The question of the stability of surfaces of separation is most closely tied in with the question of the evolution of cyclones. As yet, there is no generally accepted theory of cyclogenesis. However, out of the existing qualitative theories of cyclogenesis the most popular is the fronto-logical, or the wave, theory. At the basis of this theory is the widely known fact that almost all cyclones of the temperate and polar latitudes evolve on the giant waves generated on the surfaces of discontinuity, which separate the almost steady and recti-linear air currents.

However, by far not all wave perturbations of steady fronts are transformed into cyclones. In some cases, the wave perturbation, in passing along the front, leaves it unchanged, in other cases, the wave perturbation leads to a vortex-like motion about the front, with the front not returning to its original position, the wave becomes steeper and around its crest a field of closed isobars and an area of lower pressure, i.e. a cyclone, is evolved.

In connection with this, the theoreticians are faced with the important problem of finding quantitative criterions, which would furnish the answer, as to where and when the frontal wave evolved subsequently develops into a cyclone. The Norwegian school of meteorologists identifies the moment of the evolution of the cyclonic wave with the moment of loss of stability on the part of the frontal surface. It is obvious that the loss of stability may be tied in with the presence of mechanical factors, as well as with the presence of thermodynamic factors.

In the text immediately following only mechanical factors will be analyzed.

Numerous attempts at the determination of criterions of stability did not produce any substantial results for a long time. The main difficulty consists in that the surfaces of discontinuity are sloping surfaces, which cross the rigid surface of the earth. Hence, the problem of wave propagation on a frontal surface is a three-dimensional problem, and, therefore, a very complex one.

Some foreign theoreticians analyzed wave perturbations either on horizontal or on vertical surfaces of discontinuity. V. Bjerknes and his co-authors in his book "Physical Hydrodynamics" analyzed wave perturbations on a sloping frontal surface, separating two vertical air masses extending into infinity. All this research did not lead to any substantial results.

The first one to solve the problem of the stability of a sloping surface of separation with relation to wave motions, was N. E. Kochin, who, thereby, furnished the theoretical foundation for the wave theory of cyclogenesis.

In solving this problem by the method of small oscillations, N. E. Kochin made the allowance that, in equations of motion, the vertical accelerations may be disregarded. With the aid of clever transformations of coordinates, N. E. Kochin carried the entire complexity of boundary conditions into the equations of motion.

Let us utilize the equations of small oscillations, derived in Section 2, to the solution of the Kochin problem. As the basic motions, we will accept the established rectilinear motions of 2 liquids parallel to a surface of separation. Assuming that the densities of both liquids are constant and are, respectively, equal to ρ_1, and ρ_2. We interpolate a right-handed helical system of coordinates (Figure 151), directing axis Ox parallel to the direction of the motion, axis Oz vertically up, and axis Oy perpendicular to the front, in the direction of the cold air mass.

Figure 151. Deriving formulas for small oscillations of a surface of separation.

Under the assumptions made for the basic motions, we have:

$$\left.\begin{array}{l} U_1 = const.;\ V_1 = 0;\ W_1 = 0;\ Q_1 = \rho_1 = const.; \\ U_2 = const.;\ V_2 = 0;\ W_2 = 0;\ Q_2 = \rho_2 = const. \end{array}\right\} \qquad (1)$$

In addition:

$$X = 0;\ Y = 0;\ Z = -g. \qquad (2)$$

From equations (2) of Section 2, it follows that pressures in the warm and the cold masses for the basic motions, will be as follows:

$$\left.\begin{aligned} P_1 &= -g\rho_1 z - 2\rho_1 U_1(\omega_z y - \omega_y z) + C, \\ P_2 &= -g\rho_2 z - 2\rho_2 U_2(\omega_z y - \omega_y z) + C, \end{aligned}\right\} \qquad (3)$$

where C is the value for pressure at the origin of the coordinates. The equation of the surface of separation in the basic motion, satisfying conditionally $P_1 = P_2$, will be:

$$(\rho_2 - \rho_1) g z + 2\omega_y(\rho_1 U_1 - \rho_2 U_2) z - 2\omega_z(\rho_1 U_1 - \rho_2 U_2) y = 0, \qquad (4)$$

or

$$z = y \tan \alpha, \qquad (5)$$

where

$$\tan \alpha = \frac{2\omega_z(\rho_2 U_2 - \rho_1 U_1)}{g(\rho_1 - \rho_2) + 2\omega_y(\rho_2 U_2 - \rho_1 U_1)}. \qquad (6)$$

Consequently, the surface of separation in the basic motion is a plane, sloping to the horizon at angle α, which is determined by the Margules formula (6).

Assuming now that upon the basic motion small oscillations are superposed. Let the resulting non-steady motion be very close to the basic steady motion so that the velocity components and pressure in the warm and cold masses will be, respectively

$$U_1 + u_1,\ v_1,\ w_1,\ P_1 + p_1,$$

$$U_2 + u_2,\ v_2,\ w_2,\ P_2 + p_2,$$

and densities, as per condition above, will be constant: $\rho_1 = \text{const.}$, $\rho_2 = \text{const.}$ It being the case that u, v, w, p are infinitesimally small.

As per (13) and (14) of Section 2, for the determination of unknown functions u, v, w, p, we have equations:

$$\frac{\partial u_j}{\partial t} + U_j \frac{\partial u_j}{\partial x} = -\frac{1}{\rho_j}\frac{\partial p_j}{\partial x} - 2\omega_y w_j + 2\omega_z v_j,$$

$$\frac{\partial v_j}{\partial t} + U_j \frac{\partial v_j}{\partial x} = -\frac{1}{\rho_j}\frac{\partial p_j}{\partial y} - 2\omega_z u_j + 2\omega_x w_j, \qquad (7)$$

$$\frac{\partial w_j}{\partial t} + U_j \frac{\partial w_j}{\partial x} = -\frac{1}{\rho_j}\frac{\partial p_j}{\partial z} - 2\omega_x v_j + 2\omega_y u_j,$$

$$\frac{\partial u_j}{\partial x} + \frac{\partial v_j}{\partial y} + \frac{\partial w_j}{\partial z} = 0 \qquad (j = 1.2)$$

Assuming further that both air masses are restricted from the top and bottom by stationary horizontal planes, namely from the bottom by plane $z = 0$, and from the top by plane $z = h$. On these stationary surfaces the following conditions must be fulfilled:

$$\left.\begin{aligned} w_1 &= 0 \text{ when } z = h, \\ w_2 &= 0 \text{ when } z = 0. \end{aligned}\right\} \tag{8}$$

The equation of the surface of discontinuity in the resulting non-steady motion, we take in the form of

$$z = y \tan\alpha + \zeta(x, y, t), \tag{9}$$

where ζ is a very small magnitude. Then, for the perturbed motion, in accordance with (20) of section 2, we will have:

$$\begin{aligned} p_2 - p_1 &= P_1 - P_2 = \\ &= (\rho_2 - \rho_1) g z + 2(\rho_1 U_1 - \rho_2 U_2)\omega_y z - 2(\rho_1 U_1 - \rho_2 U_2)\omega_z y = \\ &= [(\rho_2 - \rho_1) g + 2(\rho_1 U_1 - \rho_2 U_2)\omega_y](z - y \tan\alpha) \end{aligned} \tag{10}$$

or

$$p_2 - p_1 = [(\rho_2 - \rho_1) g + 2(\rho_1 U_1 - \rho_2 U_2)\omega_y]\,\zeta \tag{11}$$

on the surface,

$$z = y \tan\alpha + \zeta(x, y, t).$$

In addition, on the same surface, as per condition (18) of Section 2, there should be:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} U_j + v_j \tan\alpha - w_j = 0 \quad (j = 1.2) \tag{12}$$

$$\text{when } z = y \tan\alpha + \zeta,$$

which is the result derived, if we assume that

$$F = y \tan \alpha - z; \quad f = f(x, y, t). \tag{13}$$

At this point, N. E. Kochin makes a substantial and daring assumption, namely, the disregarding of the vertical accelerations

$$\frac{\partial w_j}{\partial t} + U_j \frac{\partial w_j}{\partial x}$$

in the equations of motion. The disregarding of vertical accelerations is widely practiced in the study of long waves. Such disregarding is equivalent to the assumption that the vertical pressure distribution is subject to the barometric formula. Thus, p_1 and p_2 are independent of coordinate z, and are functions of only x, y, and t. By analogy, we will assume that perturbations u_1, u_2, v_1, v_2 are functions of y, y, t. The assumptions made are justified by the fact that the vertical extent of surfaces of discontinuation is many times smaller than their horizontal dimensions.

Let us find the values of w_1 and w_2 from the last equation (7):

$$\left.\begin{aligned} w_1 &= \int_z^h \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) dz = (h - z)\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_1}{\partial y}\right), \\ w_2 &= -\int_0^z \left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right) dz = -z\left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right). \end{aligned}\right\} \tag{14}$$

Equations (14) show that the vertical velocities w_j ($j=1, 2$) are small as compared to the horizontal velocities, since, by virtue of the small angle of slope, the vertical range of the motion is much smaller than the horizontal one. Therefore, in the first two equations (7), terms containing w_j may be eliminated, and, together with said terms, ω_x and ω_y will disappear from these equations. The terms containing ω_x and ω_y are eliminated by Kochin also from the third one of equations (7). Assuming, for brevity, that $\omega_z = \omega$ we derive, on the basis of assumptions made, for the determination of the seven functions u_1, v_1, u_2, v_2, p_1, p_2, ζ, the following seven equations:

$$\left.\begin{aligned} &\frac{\partial u_j}{\partial t} + U_j \frac{\partial u_j}{\partial x} = -\frac{1}{\rho_j}\frac{\partial p_j}{\partial x} + 2\omega v_j, \\ &\frac{\partial v_j}{\partial t} + U_j \frac{\partial v_j}{\partial x} = -\frac{1}{\rho_j}\frac{\partial p_j}{\partial y} - 2\omega u_j, \end{aligned}\right\} (j=1,\ 2) \qquad (15)$$

$$g(\rho_2 - \rho_1)\zeta = p_2 - p_1,$$

$$\frac{\partial \zeta}{\partial t} + U_1 \frac{\partial \zeta}{\partial x} = -v_1 \tan\alpha + (h - y\tan\alpha)\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right),$$

$$\frac{\partial \zeta}{\partial t} + U_2 \frac{\partial \zeta}{\partial x} = -v_2 \tan\alpha - y\tan\alpha\left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y}\right)$$

(into the last two equations, in the place of z was substituted $y \tan\alpha$, and magnitudes ζw_1 and ζw_2 were disregarded as infinitesimally small of the second order).

Section 7. Zonal Oscillations of a Surface of Separation.

Let us first analyze the oscillations of a surface of separation in a direction perpendicular to a front, i.e., the so-called zonal oscillations. Examples of zonal oscillations are the oscillations of a surface of separation, which has the form of a surface of rotation around the terrestrial axis. This surface separates the polar cap of cold air from the warmer tropical air. Such zonal oscillations of a surface of separation occur in a meridional plane, and are independent of longitude. In the contemplated case, not one of the seven unknown functions depend on the x-coordinate; therefore, equations (15) of the preceding section are simplified, and assume the form

$$\left.\begin{aligned} &\frac{\partial u_j}{\partial t} = 2\omega v_j \\ &\frac{\partial v_j}{\partial t} = -\frac{1}{\rho_j}\frac{\partial p_j}{\partial y} - 2\omega u_j, \\ &p_2 - p_1 = g(\rho_2 - \rho_1)\zeta, \\ &\frac{\partial \zeta}{\partial t} = -v_1 \tan\alpha + (h - y\tan\alpha)\cdot\frac{\partial v_1}{\partial y}, \\ &\frac{\partial \zeta}{\partial t} = -v_2 \tan\alpha - y\tan\alpha\,\frac{\partial v_2}{\partial y}. \end{aligned}\right\} \quad (1)$$

Proceeding to the integration of the system (1) of differential equations of the first order, we can, first of all, by differentiation, reduce the number of equations and the number of unknown functions, but at the expense of raising the order of the equations. Differentiating the third equation by y , and

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taking into account the second equation, we derive:

$$g(\rho_2-\rho_1)\frac{\partial \zeta}{\partial y}=\frac{\partial p_2}{\partial y}-\frac{\partial p_1}{\partial y}=\rho_1\frac{\partial v_1}{\partial t}-\rho_2\frac{\partial v_2}{\partial t}+2\omega(\rho_1 u_1-\rho_2 u_2). \quad (2)$$

Thus, we eliminated the unknown functions p_1 and p_2, having reduced the number of equations to five. We further eliminate two more unknown functions u_1 and u_2. Differentiating equation (2) by t and taking into account the third of equations (1), we derive, in the place of the first five of equations (1), only one equation

$$g(\rho_2-\rho_1)\frac{\partial^2 \zeta}{\partial y\,\partial t}=\rho_1\frac{\partial^2 v_2}{\partial t^2}-\rho_2\frac{\partial^2 v_2}{\partial t^2}+4\omega^2(\rho_1 v_1-\rho_2 v_2). \quad (3)$$

Finally, out of the last two of equations (1), after eliminating ζ , we derive:

$$\frac{\partial[(h-y\tan\alpha)v_1]}{\partial y}=-\frac{\partial[v_2 y\tan\alpha]}{\partial y} \quad (4)$$

or

$$(h-y\tan\alpha)v_1+y\tan\alpha\, v_2=f(t) \quad (5)$$

Let us determine the type of the function f(t). Assuming y=0, we derive:

$$h v_1(0,t)=f(t).$$

Thus, function f(t) is determined by the boundary condition, namely by the assignment of function v_1 (y, t), when f(t)=0. For simplification purposes, it can be accepted that f(t)=0.

Kochin now interpolates a new auxiliary function V(y, t), assuming that

$$V(y, t) = (h - y\tan\alpha)v_1 = -y\tan\alpha\, v_2. \qquad (6)$$

Then:

$$v_1 = \frac{V}{h - y\tan\alpha}; \quad v_2 = -\frac{V}{y\cdot\tan\alpha}, \qquad (7)$$

Whence

$$\frac{\partial^2 v_1}{\partial t^2} = \frac{\frac{\partial^2 V}{\partial t^2}}{h - y\tan\alpha}; \quad \frac{\partial^2 v_2}{\partial t^2} = -\frac{\frac{\partial^2 V}{\partial t^2}}{y\tan\alpha} \qquad (8)$$

In addition, it follows from (6) and (1) that

$$\frac{\partial V}{\partial y} = \frac{\partial \zeta}{\partial t}. \qquad (9)$$

Substituting (8) and (9) into (3), we derive:

$$g(\rho_2 - \rho_1)\frac{\partial^2 V}{\partial y^2} - \left(\frac{\rho_1}{h - y\tan\alpha} + \frac{\rho_2}{y\tan\alpha}\right)\left(\frac{\partial^2 V}{\partial t^2} + 4\omega^2 V\right) = 0. \qquad (10)$$

Thus, the system of equations (1) is reduced to one differential equation of the second order with relation to one unknown function $V(y, t)$.

We formulate the boundary conditions for this new unknown function. The surface of discontinuity, $z = y \tan\alpha$ crosses the upper stationary boundary surface $z=h$ along a straight line $y=l$; $z=h$, where $l = h \cot\alpha$ (Figure 151). Therefore, we will apply equation (10) only within the limits $0 \leqslant y \leqslant l$. At the ends of this interval, according to (6), there must be:

$$V(0, t) = V(l, t) = 0, \tag{11}$$

if only the natural assumption is made that v_1 and v_2 do not become infinity, with $y=0$ and $y=l$.

The initial conditions for V and $\frac{\partial V}{\partial t}$ we derived from (6) and (2):

$$V(y, 0) = (h - y\tan\alpha) \cdot v_1(y, 0), \tag{12}$$

$$\left(\frac{\rho_1}{h - y\tan\alpha} + \frac{\rho_2}{y\tan\alpha}\right) \cdot \frac{\partial V(y, 0)}{\partial t} =$$

$$= g(\rho_2 - \rho_1)\frac{\partial \zeta(y, 0)}{\partial y} + 2\omega\left[\rho_2 u_2(y, 0) - \rho_1 u_1(y, 0)\right].$$

It is obvious that all initial perturbations of the first mass u_1, v_1, ζ, and also the initial values of the tangential component of velocity perturbation of the second mass u_2, can be considered arbitrary.

Now then, N. E. Kochin reduced the problem of zonal oscillations of a surface of separation to the integration of an equation in individual derivatives of the second order (10), with boundary conditions as per equation (11) and initial conditions as per equation (12).

Kochin then interpolates a new dimensionless variable η, assuming that:

$$\eta = \frac{2y - l}{l}; \quad y = l\frac{\eta + 1}{2}. \tag{13}$$

After obvious transpositions, equation (10) assumes the form of

$$(1+\eta^2)\frac{\partial^2 V}{\partial \eta^2} - \frac{l(\rho_1+\rho_2)}{2g(\rho_2-\rho_1)\tan\alpha}\left[1 - \frac{\rho_2-\rho_1}{\rho_1+\rho_2}\cdot\eta\right]\left[\frac{\partial^2 V}{\partial t^2} + 4\omega^2 V\right] = 0, \tag{14}$$

and boundary conditions (11) are easily reduced to:

$$V(\pm 1, t) = 0. \tag{15}$$

Let us now, together with Kochin, effect a subsequent simplification. Within the square brackets, we disregard the fraction $\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}$, on account of its minuteness as compared to unity, and we interpolate mean density $\rho = \frac{\rho_1 + \rho_2}{2}$. Then, the differential

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equation (14) will assume the form:

$$\left(1-\eta^2\right)\frac{\partial^2 V}{\partial \eta^2}-\frac{l\rho}{g(\rho_2-\rho_1)\tan\alpha}\left[\frac{\partial^2 V}{\partial t^2}+4\omega^2 V\right]=0. \qquad (16)$$

Since we are analyzing small oscillations, it is natural to seek the solution of equation (16) by the Fourier method, i.e., in the form of the product of some periodic function of time t by some function of η. Now then, the individual solution (16) is sought in the form of

$$V(\eta,t)=e^{i\sigma t}W(\eta) \qquad (17)$$

The substitution of (17) into (16) results, for the determination of the unknown function $W(\eta)$, in a conventional differential equation of the second order

$$\left(1-\eta^2\right)\frac{d^2W}{d\eta^2}+\mu W=0, \qquad (18)$$

in which, for the sake of brevity, it is assumed that:

$$\mu=\frac{l\rho(\sigma^2-4\omega^2)}{g(\rho_2-\rho_1)\tan\alpha}. \qquad (19)$$

The boundary condition for W, as it follows from equation (15), has the form

$$W(\pm 1, t)=0. \qquad (20)$$

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Equation (18) has two particular points $\eta = +1$ and $\eta = -1$. But, applying the theory of differential equations, it can be shown that the solution of equation (18), which will satisfy the boundary condition (20), must be holomorphic (i.e. it should be expandable into an exponential series) at particular points

Differentiating equation (18) by η, we derive a differential equation for function $W'(\eta)$ as below:

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dW'}{d\eta}\right] + \mu W' = 0. \tag{21}$$

This equation has a holomorphic solution at points $\eta = \pm 1$ only for definite values of μ, namely for

$$\mu = n(n+1)$$

where $n = 0, 1, 2 \ldots$ The solution thus obtained are well known special functions, the so-called Legendre polynomials:

$$W_n'(\eta) = P_n(\eta) = \frac{1}{2^n \cdot n!} \frac{d^n(\eta^2 - 1)^n}{d\eta^n}, \tag{22}$$

Integrating (22) by η and taking into account conditions (20), we derive:

$$W_n(\eta) = \frac{1}{2^n \cdot n!} \frac{d^{n-1}(\eta^2 - 1)^n}{d\eta^{n-1}}, \tag{23}$$

with the case of $n = 0$ now to be eliminated.

As the function $W(\eta)$, any one of the polynomials below can be taken:

$$\left.\begin{aligned} W_1(\eta) &= \frac{\eta^2 - 1}{2}, \\ W_2(\eta) &= \frac{\eta(\eta^2 - 1)}{2}, \\ W_3(\eta) &= 6(\eta^2 - 1)(5\eta^2 - 1), \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{24}$$

Thus, we found a whole series of particular solutions for equation (16), which satisfy the boundary conditions (15). These particular solutions are:

$$V = W_n(\eta)\cos\sigma_n t; \quad V = W_n(\eta)\sin\sigma_n t, \tag{25}$$

it being the case that σ_n must satisfy condition

$$\frac{l\rho(\sigma_n^2 - 4\omega^2)}{g(\rho_2 - \rho_1)\tan\alpha} = n(n+1) \qquad (n = 1, 2, \ldots) \tag{26}$$

or

$$\sigma_n = \sqrt{4\omega^2 + \frac{g(\rho_2 - \rho_1)\tan\alpha\,(n+1)n}{l\rho}} \qquad (n = 1, 2 \ldots). \tag{27}$$

Taking into account the Margules formula

$$\tan\alpha = \frac{2\omega(\rho_1 U_1 - \rho_2 U_2)}{g(\rho_2 - \rho_1)},$$

we derive:

$$\sigma_n = \sqrt{4\omega^2 + \frac{2\omega(\rho_1 U_1 - \rho_2 U_2)(n+1)n}{l\rho}} . \tag{28}$$

The arrived at particular solutions of equation (16) express simple periodic oscillations of a surface of discontinuity, with period $T = \frac{2\pi}{\sigma_n}$. Assuming $\varphi = 30°$, $U_1 = -U_2 = 10$ meters sec^{-1}, $l = 10^6$ meters, we derive:

When $n=1$ $\sigma_n = 9.1 \times 10^{-5}$ sec^{-1}; $T = 0.8$ diurnal period (24 hrs.)

" $n=2$ $\sigma_n = 9.85 \times 10^{-5}$ sec^{-1}; $T = 0.74$ " " "

" $n=3$ $\sigma_n = 11.78 \times 10^{-5}$ sec^{-1}; $T = 0.62$ " " "

" $n=4$ $\sigma_n = 14.12 \times 10^{-5}$ sec^{-1}; $T = 0.52$ " " "

i.e. even the greatest period does not exceed 24 hours.

It must be emphasized that the frequency of oscillation σ_n turned out to be a substantial number. This means that the surface of separation is stable with relation to all zonal perturbations.

If the oscillation frequency turned out to be an imaginary number, we would have discovered non-stability of motion, which would have occurred in the case, when over the surface of discontinuity there is a colder mass than under the surface.

After determining the particular solutions of equation (16) by the Fourier method, it is simple to find the general solution for this equation, which satisfies the boundary conditions (10):

$$V(\eta,t)=\sum_{n=1}^{\infty}(A_n\cos\sigma_n t+B_n\sin\sigma_n t)\,W_n(\eta), \tag{29}$$

where An and Bn are arbitrary constants. Hence

$$V(\eta,0)=\sum_{n=1}^{\infty}A_nW_n(\eta);\quad \frac{\partial V(\eta,0)}{\partial t}=\sum_{n=1}^{\infty}B_n\sigma_nW_n(\eta). \tag{30}$$

But functions $V(\eta,0)$ and $\frac{\partial V(\eta,0)}{\partial t}$ must satisfy the initial conditions (12), which, in the new variables, have the form

$$V(\eta,0)=\frac{h}{2}(1-\eta)\cdot v_1(\eta,0);$$

$$\frac{1}{1-\eta^2}\frac{\partial V(\eta,0)}{\partial t}=\frac{g(\rho_2-\rho_1)\tan\alpha}{2\rho}\cdot\frac{\partial\xi(\eta,0)}{\partial\eta}+\frac{\omega h}{2}\left[u_2(\eta,0)-u_1(\eta,0)\right] \tag{31}$$

Let us also note that all polynomials $W_k(\eta)$ satisfy the condition of orthogonality:

$$\left.\begin{aligned}&\int_{-1}^{+1}\frac{W_k(\eta)W_l(\eta)}{1-\eta^2}\,d\eta=0 \quad \text{when } k\neq l\\ &\int_{-1}^{+1}\frac{W_k^2(\eta)}{1-\eta^2}\,d\eta=\frac{2}{k(k+1)(2k+1)}.\end{aligned}\right\} \tag{32}$$

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By using the correlationships (30), (31), and (32), all coefficients of series (30) can be determined. For this, it is sufficient to multiply (30) by $W_\varkappa(\eta)$, and integrate the result for η within the limits from -1 to $+1$.

Then, we derive:

$$\left.\begin{aligned} \int_{-1}^{+1} \frac{V(\eta,0)\cdot W_\varkappa(\eta)}{1-\eta^2}\,d\eta &= A_\varkappa \frac{2}{\varkappa(\varkappa+1)(2\varkappa+1)}, \\ \int_{-1}^{+1} \frac{\frac{\partial V(\eta,0)}{\partial t}\cdot W_\varkappa(\eta)}{1-\eta^2}\,d\eta &= B_\varkappa \frac{2\sigma_\varkappa}{\varkappa(\varkappa+1)(2\varkappa+1)}. \end{aligned}\right\} \qquad (33)$$

Correlationships (33) specifically determine all coefficients of Series (30), if only functions $V(\eta,0)$ and $\frac{\partial V(\eta,0)}{\partial t}$, which are expressed by formulas (31), satisfy the condition of discontinuity.

With the aid of equation

$$\frac{\partial \zeta}{\partial t} = \frac{\partial V}{\partial \eta}\cdot\frac{2}{l} \qquad (34)$$

we find function $\zeta(\eta,t)$ i.e, we determine the type of the perturbed surface of discontinuity.

$$\frac{\partial \zeta}{\partial t} = \frac{2}{l}\left\{\sum_{n=1}^{\infty}(A_n\cos\sigma_n t + B_n \sin\sigma_n t)P_n(\eta)\right\} \qquad (35)$$

whence

$$\zeta(\eta,t)=\zeta(\eta,0)+\frac{2}{l}\sum_{n=1}^{\infty}\left[A_n\sin\sigma_n t+B_n(1-\cos\sigma_n t)\right]\frac{P_n(\eta)}{\sigma_n}. \tag{36}$$

From equation (7) we find v_1 and v_2:

$$\left.\begin{aligned}
v_1(\eta,t)&=\frac{2}{1-\eta}\cdot\frac{1}{h}\sum_{n=1}^{\infty}(A_n\cos\sigma_n t+B_n\sin\sigma_n t)W_n(\eta),\\
v_2(\eta,t)&=\frac{2}{1+\eta}\cdot\frac{1}{h}\sum_{n=1}^{\infty}(A_n\cos\sigma_n t+B_n\sin\sigma_n t)W_n(\eta).
\end{aligned}\right\} \tag{37}$$

Functions u_1 and u_2 we find from equation (1):

$$\left.\begin{aligned}
&u_1(\eta,t)=u_1(\eta,0)-\\
&-\frac{2}{1-\eta}\cdot\frac{2\omega}{h}\sum_{n=1}^{\infty}\left[A_n\sin\sigma_n t+B_n(1-\cos\sigma_n t)\right]\frac{W_n(\eta)}{\sigma_n},\\
&u_2(\eta,t)=u_2(\eta,0)-\\
&-\frac{2}{1+\eta}\cdot\frac{2\omega}{h}\sum_{n=1}^{\infty}\left[A_n\sin\sigma_n t+B_n(1-\cos\sigma_n t)\right]\frac{W_n(\eta)}{\sigma_n}.
\end{aligned}\right\} \tag{38}$$

Finally, functions p_1 and p_2 are also determined from equation (1):

$$\left.\begin{aligned}
&p_1(\eta,t)=p_1(0,t)+l\rho_1\omega\int_0^{\eta}v_1(\eta,0)\,d\eta-\\
&-\frac{8\omega^2}{h}\sum_{n=1}^{\infty}\frac{1}{\sigma_n}\left[A_n\sin\sigma_n t+B_n(1-\cos\sigma_n t)\right]\int_0^{\eta}\frac{W_n(\eta)}{1-\eta}\,d\eta-\\
&-\frac{2}{h}\sum_{n=1}^{\infty}\sigma_n\left[B_n\cos\sigma_n t-A_n\sin\sigma_n t\right]\int_0^{\eta}\frac{W_n(\eta)}{1-\eta}\,d\eta;\\
&p_2(\eta,t)=p_2(0,t)+l\rho_2\omega\int_0^{\eta}v_2(\eta,0)\,d\eta-\\
&-\frac{8\omega^2}{h}\sum_{n=1}^{\infty}\frac{1}{\sigma_n}\left[A_n\sin\sigma_n t+B_n(1-\cos\sigma_n t)\right]\cdot\int_0^{\eta}\frac{W_n(\eta)}{1+\eta}\,d\eta-\\
&-\frac{2}{h}\sum_{n=1}^{\infty}\sigma_n\left[B_n\cos\sigma_n t-A_n\sin\sigma_n t\right]\cdot\int\frac{W_n(\eta)}{1+\eta}\,d\eta.
\end{aligned}\right\} \tag{39}$$

N. E. Kochin illustrates these results by the following example. Assuming that, at an initial moment t=0, the surface of separation is a sloping plane, but its angle of slope does not coincide with the angle determined by the Margules formula, but differs from it by a small magnitude. Then $\zeta(\eta, 0)$ will be a linear function of η, and it can be assumed that

$$\zeta(\eta, 0) = a\eta$$

where a is a constant. In addition, we assume that no other perturbations of the steady motions of the two given currents occur, i.e. that

$$u_j(\eta, 0) = v_j(\eta, 0) = p_j(\eta, 0) = 0 .$$

Then, by (31) we find:

$$V(\eta, 0) = 0,$$

$$\frac{\partial V(\eta, 0)}{\partial t} = \frac{g(\rho_2 - \rho_1)\tan\alpha}{2\rho} \cdot a(1-\eta^2) = -\frac{2\omega}{\rho}(\rho_1 U_1 - \rho_2 U_2) a \cdot W_1(\eta),$$

and, taking into account (30), we derive:

$$A_k = 0 \qquad (k = 1, 2, \ldots),$$

$$B_k = 0 \qquad (k = 2, 3, \ldots),$$

$$B_1 = -\frac{2\omega}{\rho\sigma_1}(\rho_1 U_1 - \rho_2 U_2) \cdot a,$$

where:

$$\sigma_1 = \sqrt{4\omega^2 + \frac{4\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho l}}.$$

Repeating the computations, conducted above for the general case, we derive:

$$V(\eta,t)=\frac{\omega(\rho_1U_1-\rho_2U_2)}{\rho\sigma_1}a\sin\sigma_1t(1-\eta^2);$$

$$v_1(\eta,t)=\frac{2\omega(\rho_1U_1-\rho_2U_2)}{\rho\sigma_1h}a(1+\eta)\sin\sigma_1t,$$

$$v_2(\eta,t)=-\frac{2\omega(\rho_1U_1-\rho_2U_2)}{\rho\sigma_1h}a(1-\eta)\sin\sigma_1t,$$

$$u_1(\eta,t)=\frac{4\omega^2(\rho_1U_1-\rho_2U_2)}{\rho\sigma_1^2h}\cdot a(1+\eta)(1-\cos\sigma_1t),$$

$$u_2(\eta,t)=-\frac{4\omega^2(\rho_1U_1-\rho_2U_2)}{\rho\sigma_1^2h}a(1-\eta)(1-\cos\sigma_1t),$$

$$p_1(\eta,t)=-\frac{4\omega^3(\rho_1U_1-\rho_2U_2)}{\sigma_1^2h}\left(\eta+\frac{\eta^2}{2}\right)\left[1+\frac{\rho_1U_1-\rho_2U_2}{\rho l\omega}\cos\sigma_1t\right],$$

$$p_2(\eta,t)=\frac{4\omega^3(\rho_1U_1-\rho_2U_2)}{\sigma_1^2h}\left(\eta-\frac{\eta^2}{2}\right)\left[1+\frac{\rho_1U_1-\rho_2U_2}{\rho l\omega}\cos\sigma_1t\right],$$

$$\zeta(\eta,t)=\frac{4\omega^2a\eta}{\sigma_1^2}\left(1+\frac{\rho_1U_1-\rho_2U_2}{\rho l\omega}\cos\sigma_1t\right).$$

The vertical velocity components can be determined by proceeding from formula (14) of the preceding Section

$$w_1=\frac{2(h-z)}{l}\cdot\frac{\partial v_1}{\partial\eta}=\frac{4\omega(\rho_1U_1-\rho_2U_2)}{\rho\sigma_1l}a\left(1-\frac{z}{h}\right)\sin\sigma_1t,$$

$$w_2=-\frac{2z}{l}\cdot\frac{\partial v_2}{\partial\eta}=-\frac{4\omega(\rho_1U_1-\rho_2U_2)}{\rho\sigma_1l}\cdot a\cdot\frac{z}{h}\sin\sigma_1t.$$

We also determine the acceleration components:

$$\frac{dv_1}{dt} = \frac{2\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho h} a(1+\eta)\cos\sigma_1 t,$$

$$\frac{dv_2}{dt} = -\frac{2\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho h} a(1-\eta)\cos\sigma_1 t,$$

$$\frac{du_1}{dt} = \frac{4\omega^2(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1 h} a(1+\eta)\sin\sigma_1 t,$$

$$\frac{du_2}{dt} = -\frac{4\omega^2(\rho_1 U_1 - \rho_2 U_2)}{\rho\sigma_1 h} a(1-\eta)\sin\sigma_1 t,$$

$$\frac{dw_1}{dt} = \frac{4\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho l} a\left(1 - \frac{z}{h}\right)\cos\sigma_1 t,$$

$$\frac{dw_2}{dt} = -\frac{4\omega(\rho_1 U_1 - \rho_2 U_2)}{\rho l} a \cdot \frac{z}{h}\sin\sigma_1 t.$$

The analysis of the solution obtained makes it possible to draw the following conclusions:

(1) The surface of discontinuity, the slope of which does not conform to the Margules formula, does not remain steady. It oscillates about some mean position, which is determined by equality

$$\zeta_0 = \frac{4\omega^2}{\sigma_1^2} \cdot a\eta.$$

This mean position does not coincide with the position of the steady surface of separation, and is departed from the latter in the same direction as during the initial moment.

(2) The tangential components of the perturbed velocities u_1 and u_2, oscillating in magnitude, retain at all times the same algebraic sign. When $a > 0$, $u_1 > 0$, and $u_2 < 0$. Consequently, if the surface of discontinuity had a steeper slope than the steady surface, the velocity of the first basic current U_1, will somewhat increase in its mean value, and the velocity of the second basic current U_2 will somewhat decrease in its mean value. Consequently, the discontinuity in velocity at the surface of discontinuity will somewhat increase in its mean value. This increase in the discontinuity of the tangential velocity component will result, as per the Margules formula, in a higher value of the angle α.

(3) The derived solution shows that in the same manner as the increase of discontinuity in velocity leads to the increase in the angle of slope, the increase of the angle of slope leads to an increase in the discontinuity of velocity.

(4) Non-steady motions which are generated at the surface of separation and are close to the basic motion, turn out to be very complex, and here are evolved additional velocities in all directions.

Let us clarify in more detail as to what will occur if at the initial moment the surface of separation was rising more steeply than the steady surface so that $\alpha > 0$, and there are no other perturbations of the basic motion. The formulas derived in this Section show that specifically then the underflow of the cold air mass under the warm air mass begins. It is accompanied by a descending motion of the cold air which motion is particularly intensive in its upper part, and an ascensional motion of the warm air. Thus, in the upper part the warm air will displace the cold air, and the slope of the surface of separation will become shallower.

As soon as velocities v_1 and v_2 appear, the Coriolis force tends to deflect them to the right, with relation to the pressure gradients, which induced these additional velocities. Therefore, horizontal velocity components u_1 and u_2 will appear.

However, such accretion of velocity components cannot be unlimited. The moment arrives when the process is reversed. This moment corresponds to the minimum slope of the discontinuity surface.

The simplifications used [in the mathematical argumentations] made it possible for Kochin to intercept the basic configurations of the processes, which take place about a non-steady surface of separation in the case, when the latter becomes subject to zonal oscillations. However, the factual processes are considerably more complex than the diagram that was analyzed.

Let us point to the two basic factors which complicate the zonal oscillations of the non-steady surface of separation.

Until now it was assumed that the oscillations of a surface of separation are zonal, i.e. the motion of the air is perfectly the same in all planes, which are perpendicular to the front. In reality, a surface of separation is subject to non-zonal oscillations. In addition, it is subject to deformation. Such oscillations will be analyzed in the following section.

The second factor distorting the above picture of zonal oscillations is friction. The effect of friction is such that the uncontaminated oscillation of the discontinuity surface becomes impossible.

The periodic process becomes considerably less pronounced, and, in its place, an aperiodic process sets in.

Section 8. Stability of a Surface of Separation with Relation to Non-Zonal Oscillations.

In the preceding Section, the problem of zonal oscillations of a surface of separation was analyzed and it was shown that, in the case of zonal oscillations, the surface of separation cannot lose stability. We will now analyze a more general case of non-zonal oscillations of a surface of separation, which case was also analyzed by N. E. Kochin.

The initial equations can be taken in a form analogous to the form of equations (1) of the preceding paragraph:

$$\frac{\partial u_j'}{\partial t} + U_j \frac{\partial u_j'}{\partial x} = -\frac{1}{\rho}\frac{\partial p_j'}{\partial x} + 2\omega v_j' \quad (j=1,2),$$

$$\frac{\partial v_j'}{\partial t} + U_j \frac{\partial v_j'}{\partial x} = -\frac{1}{\rho}\frac{\partial p_j'}{\partial y} - 2\omega u_j' \quad (j=1,2), \qquad (1)$$

$$g(\rho_2-\rho_1)\zeta' = p_2' - p_1',$$

$$\frac{\partial \zeta'}{\partial t} + U_1 \frac{\partial \zeta'}{\partial x} = -v_1' \tan\alpha + (h - y\tan\alpha)\left(\frac{\partial u_1'}{\partial x} + \frac{\partial v_1'}{\partial y}\right),$$

$$\frac{\partial \zeta'}{\partial t} + U_2 \frac{\partial \zeta'}{\partial x} = -v_2' \tan\alpha - y\tan\alpha\left(\frac{\partial u_2'}{\partial x} + \frac{\partial v_2'}{\partial y}\right).$$

In the first equations it is assumed that $\rho \approx \rho_1 \approx \rho_2$; ω is the vertical component of the angular velocity of the earth's rotation. Function $\zeta'(x,y,t)$ expresses the perturbation of the discontinuity surface. The equation of the discontinuity surface in perturbed motion has the form of

$$z = y\tan\alpha + \zeta'(x,y,t). \qquad (2)$$

N. E. Kochin considers the waves propagating along the front of the discontinuity surface, i.e. along the Ox axis. In conformance with this, it is natural to seek the solution of the system of 7 equations (1), containing 7 unknown functions u_1', u_2', v_1', v_2', p_1', p_2', ζ', in the form:

$$\left.\begin{aligned} u_j' &= e^{i(\kappa x+\sigma' t)} u_j(y), \\ v_j' &= e^{i(\kappa x+\sigma' t)} v_j(y), \\ p_j' &= e^{i(\kappa x+\sigma' t)} p_j(y) \\ \zeta' &= e^{i(\kappa x+\sigma' t)} \zeta_j(y). \end{aligned}\right\} \qquad (3)$$

In formulas (3) functions u_j, v_j, p_j, ζ are complex. However, later on for u'_j, v'_j, p'_j, ζ, we will take the substantial part only.

We substitute (3) into the first two equations (1). After obvious reductions, we derive:

$$\left.\begin{aligned} i(\kappa U_j+\sigma')u_j-2\omega\cdot v_j&=-\frac{i\kappa}{\rho}\cdot p_j\\ i(\kappa U_j+\sigma')v_j+2\omega\cdot u_j&=-\frac{1}{\rho}\frac{dp_j}{dy},\end{aligned}\right\}\qquad(4)$$

whence

$$\left.\begin{aligned} u_j&=\frac{\kappa(\kappa U_j+\sigma')p_j-2\omega\dfrac{dp_j}{dy}}{\rho[4\omega^2-(\kappa U_j+\sigma')^2]}\\ v_j&=i\,\frac{2\omega\kappa p_j-(\kappa U_j+\sigma')\dfrac{dp_j}{dy}}{\rho[4\omega^2-(\kappa U_j+\sigma')^2]}\end{aligned}\right\}\qquad(5)$$

Eliminating u'_j, v'_j and ζ from equations (1), we derive the following system of differential equations for two functions p_1 and p_2:

$$\left.\begin{aligned} &\frac{d}{dy}\left(y\frac{dp_2}{dy}\right)-\left(\kappa^2y+\frac{2\kappa\omega}{\kappa U_2+\sigma'}\right)\cdot p_2=\\ &\qquad=\frac{\rho[4\omega^2-(\kappa U_2+\sigma')^2]}{g(\rho_1-\rho_2)\tan\alpha}\cdot(p_1-p_2),\\ &\frac{d}{dy}\left[(y-\ell)\frac{dp_1}{dy}\right]-\left[\kappa^2(y-\ell)+\frac{2\kappa\omega}{\kappa U_1+\sigma'}\right]p_1=\\ &\qquad=\frac{\rho[4\omega^2-(\kappa U_1+\sigma')^2]}{g(\rho_1-\rho_2)\tan\alpha}\cdot(p_1-p_2).\end{aligned}\right\}\qquad(6)$$

Points $y = 0$ and $y = l$ are particular points for this system of equations. We specify that at these points the solution is to remain finite. Then p_1 and p_2 will be entire functions of y.

Upon the solution of system (6), N. E. Kochin superposes the following boundary conditions:

$$\frac{dp_1}{dy} = \kappa p_1 \quad \text{when } y = 0, \tag{7}$$

$$\frac{dp_2}{dy} = -\kappa p_2 \quad \text{when } y = l. \tag{8}$$

In so doing he proceeds from the following considerations. The motion in the area $y < 0$ is analyzed. From the equation of continuity it follows that

$$w_1' = -z\left(\frac{\partial u_1'}{\partial x} + \frac{\partial v_1'}{\partial y}\right),$$

since, when $z = 0$, $w_1' = 0$. But in the area $y < 0$ with $z = h$ also $w_1' = 0$. Consequently, in general:

$$\frac{\partial u_1'}{\partial x} + \frac{\partial v_1'}{\partial y} = 0.$$

Taking into account (3), we derive:

$$i\kappa u_1 + \frac{dv_1}{dy} = 0 \quad \text{when } y < 0.$$

Substituting into this equality the values of u_1 and v_1 from (5), we derive:

$$\frac{d^2 p_1}{dy^2} - \kappa^2 p_1 = 0.$$

The general integral of this differential equation has the form of

$$p_1 = C_0 e^{\kappa y} + C_1 e^{-\kappa y},$$

it being necessary to assume that $C_1 = 0$ since with $y \to -\infty$, p_1 must remain finite. Consequently,

$$p_1 = C_0 e^{\kappa y} \text{ when } y < 0,$$

whence

$$\frac{dp_1}{dy} = \kappa p_1 \text{ when } y < 0. \qquad (9)$$

Obviously, in passing from region I to region IV (Figure 151), u_1 and v_1 must change continuously. Consequently, in accordance with (5), also functions p_1 and $\frac{dp_1}{dy}$ must change continuously. Therefore, condition (9) must be satisfied also with $y = 0$. Thus, upon the solution of system (6) is superposed a boundary condition (7). In the same manner, the second boundary condition (8) is established.

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We again interpolate a dimensionless variable

$$\eta = \frac{2y-\ell}{\ell}; \; y = \ell\frac{1+\eta}{2}. \tag{10}$$

Then, the interval $0 \leqslant y \leqslant \ell$ will be replaced by interval $-1 \leqslant \eta \leqslant +1$.

Supposing that in addition,

$$U = \frac{U_1 - U_2}{2}. \tag{11}$$

In this case, the Margules formula can be taken in the following form

$$\tan \alpha = \frac{2\omega(\rho_1 U_1 - \rho_2 U_2)}{g(\rho_2 - \rho_1)} \approx \frac{4\omega\rho U}{g(\rho_2 - \rho_1)}. \tag{12}$$

Finally, we interpolate, in place of frequency σ' a new magnitude σ :

$$\sigma = \sigma' + \frac{\kappa(U_1 + U_2)}{2} \tag{13}$$

It is obvious that

$$\sigma' + \kappa U_1 = \sigma + \kappa U, \quad \sigma' + \kappa U_2 = \sigma - \kappa U, \tag{14}$$

and the system of equations (6) assumes the form of

$$\left.\begin{aligned}
&\frac{d}{d\eta}\left[(1+\eta)\frac{dp_2}{d\eta}\right]-\left[\frac{\kappa^2 l^2}{4}(1+\eta)+\frac{\kappa\omega l}{\sigma-\kappa U}\right]p_2=\\
&\qquad=\frac{l[4\omega^2-(\sigma-\kappa U)^2]}{8\omega U}(p_2-p_1),\\
&\frac{d}{d\eta}\left[(1-\eta)\frac{dp_1}{d\eta}\right]-\left[\frac{\kappa^2 l^2}{4}(1-\eta)-\frac{\kappa\omega l}{\sigma+\kappa U}\right]p_1=\\
&\qquad=\frac{l[4\omega^2-(\sigma+\kappa U)^2]}{8\omega U}(p_1-p_2),
\end{aligned}\right\} \quad (15)$$

and the boundary conditions will assume the form

$$\left.\begin{aligned}
&\frac{dp_1}{d\eta}=\frac{\kappa l}{2}\cdot p_1 \text{ when } \eta=-1,\\
&\frac{dp_2}{d\eta}=-\frac{\kappa l}{2}\cdot p_2 \text{ when } \eta=+1.
\end{aligned}\right\} \quad (16)$$

Into equations (15) enter three dimensionless parameters:

$$\frac{\kappa U}{2\omega}=\beta;\ \frac{l\omega}{U}=\alpha;\ \frac{\sigma}{\kappa U}=\tau. \quad (17)$$

Interpolating these explicitly into equations (15), we will finally derive the following equations:

$$\left.\begin{aligned}
&\frac{d}{d\eta}\left[(1+\eta)\frac{dp_2}{d\eta}\right]-\left[\alpha^2\beta^2(1+\eta)+\frac{\alpha}{\tau-1}\right]p_2=\\
&\qquad=\frac{\alpha}{2}\left[1-\beta^2(\tau-1)^2\right](p_2-p_1),\\
&\frac{d}{d\eta}\left[(1-\eta)\frac{dp_1}{d\eta}\right]-\left[\alpha^2\beta^2(1-\eta)-\frac{\alpha}{\tau+1}\right]p_1=\\
&\qquad=\frac{\alpha}{2}\left[1-\beta^2(\tau+1)^2\right](p_1-p_2).
\end{aligned}\right\} \quad (18)$$

The boundary conditions in this case are to be taken in the form of

$$\left.\begin{aligned} &\frac{dp_1}{d\eta} - \alpha\beta p_1 = 0 \text{ when } \eta = -1, \\ &\frac{dp_2}{d\eta} + \alpha\beta p_2 = 0 \text{ when } \eta = +1. \end{aligned}\right\} \qquad (19)$$

Now then, N. E. Kočhin reduced the problem of non-zonal perturbations of a discontinuity surface to the analysis of a system of equations (18). The dimensionless parameters α, β and τ entering into these equations, have the following physical meaning: Parameter

$$\alpha = \frac{l\omega}{U} = \frac{2h\omega\cot\alpha}{U_1 - U_2} \qquad (20)$$

is determined by the given elements of the basic motion.

Parameter

$$\beta = \frac{\kappa U}{2\omega} = \frac{\pi U}{\lambda\omega} = \frac{\pi(U_1 - U_2)}{2\omega\lambda} \qquad (21)$$

characterizes the length of the waves running along the front.

Finally, parameter

$$\tau = \frac{\sigma}{\kappa U} = \frac{\sigma' + \frac{\kappa(U_1 + U_2)}{2}}{\kappa\frac{(U_1 - U_2)}{2}} \qquad (21')$$

characterizes the frequency of oscillations, which frequency is unknown and is to be determined from the contemplated equations.

Thus, with the given values of α and β it is requisite to determine those values of τ, at which the solution of system (18) satisfies the boundary conditions (19), it being the case that functions $p_1(\eta)$ and $p_2(\eta)$ must be holomorphic at points $\eta = +1$ and $\eta = -1$, and consequently must be entire functions.

In connection with the problem of cyclogenesis, it is necessary to analyze the question about the loss of stability by a surface of separation with relation to very long waves, when the value of β is very small, and consequently the length of the waves λ is great as compared to $\frac{U}{\omega}$.

N. E. Kochin underscores that at first it may seem that the case of very long waves may be reduced to the case of infinitely long waves that was analyzed in the preceding Section, when the perturbations do not depend on coordinate x, and the Margules surface of separation undergoes zonal oscillations. But with such perturbations, closely resembling the above analyzed zonal oscillations, σ has a finite value, and consequently

$$\tau = \frac{\sigma}{\varkappa U} = \frac{\sigma}{2\omega\beta}$$

has, when the values of β are small, a very great value.

Kochin pointed out that in addition to perturbations closely resembling the zonal oscillations there exists one more variety of long wave perturbations for which the magnitudes of τ retain finite values. Frequencies σ (and, consequently, also σ') of these perturbations are very small, and their periods are very great.

Kochin analyzed an ultimate case for these perturbations assuming for equations (18) and for boundary conditions (19), that $\beta = 0$.

For this ultimate case we derive:

$$\left.\begin{aligned} \frac{d}{d\eta}\left[(1+\eta)\frac{dp_2}{d\eta}\right] - \frac{\alpha}{\tau-1}\cdot p_2 &= \frac{\alpha}{2}(p_2-p_1), \\ \frac{d}{d\eta}\left[(1-\eta)\frac{dp_1}{d\eta}\right] + \frac{\alpha}{\tau+1}\cdot p_1 &= \frac{\alpha}{2}(p_1-p_2), \end{aligned}\right\} \qquad (22)$$

$$\left.\begin{aligned} \frac{dp_1}{d\eta} &= 0 \quad when \quad \eta = -1, \\ \frac{dp_2}{d\eta} &= 0 \quad when \quad \eta = +1 \end{aligned}\right\} \qquad (23)$$

Let us add up equations (22), having preliminarily multiplied the first one by $(\tau-1)$, and the second by $(\tau+1)$. The derived equation, upon integration, assumes the form of

$$(\tau-1)(1+\eta)\frac{dp_2}{d\eta} + (\tau+1)(1-\eta)\frac{dp_1}{d\eta} = const. = 0. \qquad (24)$$

The selection of a constant in equation (24) is to be made in conformity with the boundary condition (23).

Taking an auxiliary function

$$\Phi(\eta)=(1-\tau)(1+\eta)\frac{dp_2}{d\eta}=(1+\tau)(1-\eta)\frac{dp_1}{d\eta} \tag{25}$$

and interpolating it into equations (22), we derive the differential equation

$$\Phi''(\eta)=\frac{\alpha}{2}\left[\frac{\tau+1}{(\tau-1)(1+\eta)}+\frac{\tau-1}{(\tau+1)(1-\eta)}\right]\Phi(\eta), \tag{26}$$

from which $\Phi(\eta)$ can be determined. The boundary condition for the function $\Phi(\eta)$ has the form

$$\Phi(\pm 1)=0. \tag{27}$$

Equation (26) can be reduced to

$$(1-\eta^2)\Phi''(\eta)+(r+s\eta)\Phi(\eta)=0, \tag{28}$$

where

$$r=\alpha\frac{1+\tau^2}{1-\tau^2}; \quad s=-\frac{2\alpha\tau}{1-\tau^2} \tag{29}$$

and, consequently,

$$\tau=-\frac{s}{r+\alpha}, \tag{30}$$

it being the case that always

$$r^2 - s^2 = d^2. \tag{31}$$

Equation (28) has the same form as equation (21) of the preceding Section. Therefore, the solution of equation (28) is sought in the form of an expansion into a polynomial W_n series which is the integral of equation (21) of Section 7:

$$\Phi(\eta) = a_1 W_1(\eta) + a_2 W_2(\eta) + \cdots, \tag{32}$$

with the polynomials linked by the following recurrent formula:

$$(2n+1)\cdot\eta W_n = (n-1) W_{n-1} + (n+2) W_{n+1} \quad (n = 1, 2, \ldots). \tag{33}$$

Substituting (32) into equation (28), and taking into account (33) and (34),

$$(1-\eta^2) W_n'' + n(n+1) W_n = 0 \qquad (n = 1, 2, \ldots), \tag{34}$$

we derive the following correlationships between the coefficients:

$$\begin{aligned} &a_1[r - 1\cdot 2] + \frac{1}{5} s a_2 = 0, \\ &a_2[r - 2\cdot 3] + \frac{2}{7} s a_3 + \frac{3}{3} s\cdot a_1 = 0, \end{aligned} \tag{35}$$

. . . .

$$a_\kappa[r - \kappa(\kappa+1)] + \frac{\kappa}{2\kappa+3}\cdot s a_{\kappa+1} + \frac{\kappa+1}{2\kappa-1}\cdot s\cdot a_{\kappa-1} = 0$$

$$(\kappa = 2, 3, \ldots).$$

From this we derive

$$\frac{a_1}{a_2} = \frac{\frac{1}{5} \cdot s}{r - 1.2},$$

$$s \cdot \frac{a_1}{a_2} = 2 \cdot 3 - r - \frac{\delta_2 \cdot s^2}{\frac{4}{5} s \cdot \frac{a_2}{a_3}}, \tag{36}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{k+1}{2k-1} \cdot s \frac{a_{k-1}}{a_k} = k(k+1) - r - \frac{\delta_k s^2}{\frac{k+2}{2k+1} s \cdot \frac{a_k}{a_{k+1}}},$$

where

$$\delta_k = \frac{k(k+2)}{(2k+1)(2k+3)}. \tag{37}$$

By comparing the two expressions for $\frac{a_1}{a_2}$ from equation (36) we derive the following correlationship between r and S, expressed by the nonterminating continued fraction:

$$\frac{\delta_1 \cdot s^2}{1 \cdot 2 - r} = 2 \cdot 3 - r - \cfrac{\delta_2 \cdot s^2}{3 \cdot 4 - r - \cfrac{\delta_3 \cdot s^2}{4 \cdot 5 - r - \ldots}}. \tag{38}$$

Correlationships (38) and (31) fully determine r and S and, therefore, according to (30), they also determine τ. When $S = 0$ we derive from (38):

$$r_1 = 1 \cdot 2; \; r_2 = 2 \cdot 3 \ldots \; r_n = n(n+1),$$

and from (31):

$$r = \alpha.$$

Consequently, if $\alpha = n(n+1)$, one of the solutions of system (38) and (31) is:

$$r = n(n+1); \quad s = 0. \tag{39}$$

In the particular case, when $\alpha = 2$

$$r = 2 \quad \text{and} \quad s = 0.$$

Kochin contemplates values for α, close to the value of 2. If $r \approx 2$ and $s \approx 0$, we derive from (38)

$$r \approx 2 - \frac{\delta_1 s^2}{4},$$

and from (31)

$$4 - \delta_1 s^2 - s^2 + \ldots = \alpha^2$$

or, since

$$\delta_1 = \frac{1}{5},$$

then

$$4 - \frac{6}{5} s^2 + \ldots = \alpha^2.$$

Consequently,

$$s^2 \approx \frac{5}{6}(4 - \alpha^2). \tag{40}$$

From (40) it follows that if α is somewhat greater than 2, $s^2 < 0$, and s is a purely imaginary number, and, consequently, the motion will be an unstable one.

Now then, when $\alpha > 2$, the Margules motion is unstable with relation to waves of very great length. Kochin also proved that in the reverse case when $0 < \alpha < 2$, the Margules motion is a stable one with relation to waves of very great length.

We now write down the condition of stability in its explicit form, substituting into it the derived values of α:

$$\alpha = \frac{l\omega}{U} = \frac{2h\omega \cot \alpha}{U_1 - U_2} < 2.$$

Taking into account (12) the condition of stability of the Margules surface of separation, with relation to waves of very great length, can be presented in the form of

$$U > \sqrt{\frac{gh(\rho_2 - \rho_1)}{8\rho}}. \tag{41}$$

Now then, the Margules surface of separation is the more stable with relation to waves of great length, the greater the relative velocity of the basic currents and the smaller the depth of these currents, and also the smaller the discontinuity of density.

When h = 8 kilometers, $\frac{\rho_1 - \rho_2}{4\rho} = \frac{1}{40}$ (i.e. with a discontinuity of temperature $\approx 7°$), the basic motion is stable, with $U >$ 16 meters/second.

A similar result was derived by Kochin for waves of finite length. To each value of β corresponds its critical value $\alpha_0(\beta)$. With $0<\alpha<\alpha_0(\beta)$, the basic motion is stable with relation to waves having a length $\lambda = \frac{\pi U}{\omega \beta}$, and unstable with $\alpha > \alpha_0(\beta)$.

Figure 152 shows a curve $\alpha_0(\beta)$, which separates a region of stability from a region of non-stability. From this Figure it can be seen that frontal surfaces are unstable with relation to waves of small length (value of β is great). However, with relation to waves of greater length, the frontal surfaces turn out to be stable. But with relation to waves of very great length (on the order of 500-1000 kilometers), frontal surfaces again become unstable. Specifically, these waves of very great length lead to the formation of cyclones.

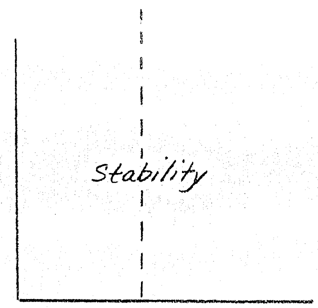

Figure 152. Dependence of the stability of a surface of separation on the length of wave.

Thus, by analyzing the schematic case above, Kochin proved the full possibility for the existence in the atmosphere of conditions, under which the long wave perturbations on a surface of separation are unstable. The presence of such non-stability results in that the surface of separation undergoes deformation, bulging, tapering, as a result of which there occurs the development of vortices, which can be identified with the cyclones observed in the atmosphere.

Let us note that the schematic features of Kochin's approach to the problem led to some objections on the part of certain foreign meteorologists. Godske maintained that, in disregarding vertical accelerations, Kochin [inadvertently] eliminated a certain type of waves. However, when, in 1935, Blinova solved the problem of zonal perturbations of a surface of separation, without disregarding the vertical accelerations, the results arrived at were very close to those derived by Kochin. This, best of all, proves the rationality of the simplifications resorted to by Kochin.

The work performed by Kochin gave impetus to a series of investigations of the same problem under more general conditions. Dorodnitsin (1936) generalized upon Kochin's zonal oscillations with relation to a barotropic medium. Yudin (1937), and then, by using another method, Blinova (1939) solved the problem of non-zonal oscillations of a surface of separation for baroclinic liquids.

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