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BASIC HYDRODYNAMIC EQUATIONS FOR TURBULENT MOTION AND THEIR APPLICATION TO MOTION IN ROUND PIPES

Stjepan Mohorovicic

(Zagreb, Yugoslavia).

/Note: The following is an article that appeared in the journal Zeitschrift fur technische Physik, pages 68-74 (1925)./

Contents:

It is well known that present-day hydrodynamics with the aid of Eulerian and Navier-Stokes theory govern only "laminar" motions. As soon as, however, the velocity exceeds in actuality a limiting value, an entirely irregular turbulent motion immediately ensues. It is assumed, nevertheless, that the basic hydrodynamic equations do not lose their validity in also this case, although the theory here is undoubtedly in need of broadening. Previously several attempts were made along these lines to expand the theory, and the most noteworthy attempts have been performed recently by J. Boussinesq [2]) and O. Reynolds [3]).

Several years ago I [4]) had made an attempt, during an investigation of the structure of wind - which is reported in more detail in another place-, to treat turbulent flow theoretically, and during it I had obtained rather good agreement with the results of my experimental measurements. It is my intention to include here generally my expansion of the theory, and to apply it to turbulent flow in round pipes.

Let us proceed from the familiar equations for an incompressible **fluid:**

$$\begin{cases} \rho \dfrac{Du}{Dt} = \rho(X + \mathfrak{X}) - \dfrac{\partial p}{\partial x} + \varepsilon \nabla^2 u \\ \rho \dfrac{Dv}{Dt} = \rho(Y + \mathfrak{Y}) - \dfrac{\partial p}{\partial y} + \varepsilon \nabla^2 v \\ \rho \dfrac{Dw}{Dt} = \rho(Z + \mathfrak{Z}) - \dfrac{\partial p}{\partial z} + \varepsilon \nabla^2 w \end{cases} \qquad (1_1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1_2)$$

where we have introduced in addition the internal forces $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ which act to retard the turbulent motion because of the large energy losses; epsilon ε is the "virtual" internal friction or turbulence [5]. Since the fluid motion is composed of two parts, we shall divide according to Reynolds [6] the velocities into two components:

$$\cancel{\begin{matrix} \mu = \bar{\mu} + \mu' \\ v = \bar{v} + v' \\ w = \bar{w} + w' \end{matrix}} \qquad \begin{cases} u = \bar{u} + u' \\ v = \bar{v} + v' \\ w = \bar{w} + w' \end{cases} \qquad (2)$$

where we disignate by $\bar{\mu}$, $\bar{v}$, $\bar{w}$ the average (mean) velocities and by μ', v', w' the components of turbulence (pulsation). **Further** we shall assume that a part of the velocity drop causes the turbulent motions, irrespective of whether primary or secondary; that is, a part of the velocity drop causes the average flow and the other part causes the turbulence. If $\bar{\theta}_i$ and $\bar{\theta}_i'$, mean the corresponding fractions of the velocity drop, we must have

$$\bar{\theta}_i + \bar{\theta}_i' = 1 \qquad (i = 1, 2, 3) \qquad (3)$$

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the basic equations $(1_{1,\,2})$ then break up into the two systems:

$$\left\{\begin{aligned} \rho\frac{D\bar{u}}{Dt} &= \rho X - \bar{\theta}_1\frac{\partial p}{\partial x} + \varepsilon\nabla^2\bar{u} \\ \rho\frac{D\bar{v}}{Dt} &= \rho Y - \bar{\theta}_2\frac{\partial p}{\partial y} + \varepsilon\nabla^2\bar{v} \\ \rho\frac{D\bar{w}}{Dt} &= \rho Z - \bar{\theta}_3\frac{\partial p}{\partial z} + \varepsilon\nabla^2\bar{w} \end{aligned}\right. \qquad (4_1)$$

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z} = 0; \qquad (4_2)$$

and

$$\left\{\begin{aligned} &\rho\left[\frac{Du'}{Dt} + \bar{u}\frac{\partial u'}{\partial x} + \bar{v}\frac{\partial u'}{\partial y} + \bar{w}\frac{\partial u'}{\partial z} + \bar{u}_x u' + \bar{u}_y v' + \bar{u}_z w'\right] = \\ &\qquad\qquad \rho\mathfrak{X} - \theta_1'\frac{\partial p}{\partial x} + \varepsilon\nabla^2 u', \\ &\rho\left[\frac{Dv'}{Dt} + \bar{u}\frac{\partial v'}{\partial x} + \bar{v}\frac{\partial v'}{\partial y} + \bar{w}\frac{\partial v'}{\partial z} + \bar{v}_x u' + \bar{v}_y v' + \bar{v}_z w'\right] = \\ &\qquad\qquad \rho\mathfrak{Y} - \theta_2'\frac{\partial p}{\partial y} + \varepsilon\nabla^2 v', \\ &\rho\left[\frac{Dw'}{Dt} + \bar{u}\frac{\partial w'}{\partial x} + \bar{v}\frac{\partial w'}{\partial y} + \bar{w}\frac{\partial w'}{\partial z} + \bar{w}_x u' + \bar{w}_y v' + \bar{w}_z w'\right] = \\ &\qquad\qquad \rho\mathfrak{Z} - \theta_3'\frac{\partial p}{\partial z} + \varepsilon\nabla^2 w', \end{aligned}\right. \qquad (5_1)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0\,. \qquad (5_2)$$

Now we must first solve the first system ($4_{1,\ 2}$) and set the values found for $\overline{u}$, $\overline{v}$, $\overline{w}$ into the second system (5_1); only then can we calculate the turbulence velocities u', v', w', which in many cases will be superfluous. The very complicated system ($5_{1,\ 2}$) makes us suspect that the pulsations (turbulence) will be of a very complicated nature, which observation also has confirmed.

I. The utility of the expansion developed here of the theory I have already indicated earlier in the investigation of the structure of wind (see footnote 4); I will now apply this theory to the case of turbulent flow in around pipes. In this case we shall first assume that the average velocities $\overline{w}$ are stationary; the z-axis coincides with the cylinder's axis; and the average velocities $\overline{w}$ are distributed around the cylinder's axis symmetrically - that is, the average velocity $\overline{w}$ is only a function of the distance r from the z-axis. Since $\overline{u} = \overline{v} = 0$, $\partial\overline{w} / \partial t = 0$, $\partial\overline{w} / \partial z = 0$, $X = Y = Z = 0$ (that is, we shall also be able to neglect gravity in the case where the pipe is in a horizontal position), we obtain from (4_1) the familar basic relation:

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial\overline{w}}{\partial r}\right) = \frac{\partial p}{\varepsilon\partial z}\,\overline{\theta}_3 \quad . \tag{6}$$

Obviously we must make assumptions for $\overline{\theta}_3$ such that they not only lead us to correct results but also correspond to the nature of our problem.

First let us set:

$$\overline{\theta}_3 = A(b + ar^n), \tag{7}$$

where A, b, a also can be functions of the pipe's radius.

In order to solve our basic equation (6), let us set:

$$\bar{w} = K_o \theta_o + B, \tag{8}$$

where K_o and B too are constants to be determined, and $\bar{\theta}_o$ should be a function of r. From (8) and (6) we obtain:

$$\theta_o'' + \theta_0'/r = K\theta_3/K_o, \tag{9}$$

where we have set

$$K = op/\ \varepsilon\ oz \tag{10}$$

In order to be able to solve the differential equation (9) with regard to (7), we set:

$$\theta_o = \kappa \cdot Aar^{n+2} + Cr^2 \tag{11}$$

and finally obtain from it

$$C = KAb/4K_o \tag{12$_1$}$$

$$n = \cancel{-2 \pm \sqrt{(K/\kappa K_o)}} \qquad n = -2 \pm \sqrt{\frac{K}{\kappa K_o}} \tag{12$_2$}$$

In the case where m is any real number, then we shall have for

$$K = m^2 \kappa K_o$$

at once $n = -2 + m$ (13)

or $n = -2 - m,$ (14)

and we see immediately that the negative values of n do not correspond to our problem. Therefore it follows that:

$$\bar{w} = \frac{KAa}{m^2} r^m + \frac{KAb}{4} r^2 + B \quad (m \geqslant 2). \tag{15}$$

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Since at the wall of the pipe ($r = R$) no slip occurs, that is since we must have $\bar{w} = 0$ there, we finally get:

$$\bar{w} = -\frac{A}{\varepsilon} \cdot \frac{\partial p}{\partial z} \left[\frac{a}{m^2} (R^m - r^m) + \frac{b}{4} (R^2 - r^2) \right] \quad (m \geqq 2) \tag{15_a}$$

For the total flow through a cross-section in 1 second we get:

$$\bar{Q}_o = \frac{A}{\varepsilon} \cdot \frac{\partial p}{\partial z} \left[\frac{m}{m+2} \cdot \frac{a\pi}{m^2} R^{m+2} + \frac{b\pi}{8} R^4 \right] \quad (m \geqq 2), \tag{16}$$

and we form the average velocity: $\bar{c} = -\bar{Q}/R^2\pi$; (17)

thus we obtain for the velocity distribution in the cross-section the expression:

$$\frac{\bar{w}}{\bar{c}} = \frac{\frac{a}{m^2}(R^m - r^m) + \frac{b}{4}(R^2 - r^2)}{\frac{m}{m+2} \cdot \frac{a}{m^2} R^m + \frac{b}{8} R^2} \tag{18}$$

It will now be interesting to consider some special cases:

1. For $A = b = 1$ and $a = 0$, our relations (15a) to (18) reduce to the familiar Poiseuille Law for laminar flow:

$$w = -\frac{1}{4\mu} \cdot \frac{\partial p}{\partial z} (R^2 - r^2) \tag{19_1}$$

$$Q_o = \frac{\pi R^4}{8\mu} \cdot \frac{\partial p}{\partial z} \qquad (\varepsilon = \mu) \tag{19_2}$$

$$\frac{w}{c} = 2 \left(1 - \frac{r^2}{R^2}\right) \tag{19_3}$$

Therefore it follows that laminar flow represents an entirely special case of natural flow. As is well known, this law loses its validity at once if the flow becomes turbulent. It follows from (19_3) that for $r = 0$ we get immediately:

$$w_o = 2c; \tag{19_4}$$

on the other hand, measurements of turbulent flows yield [8]:

$$\bar{w}_0 = 1.16\bar{c} \quad \text{up to } 1.23\bar{c} \tag{20}$$

2. For $b = 0$ and $A \cdot a = a_0$, we have already a kind of turbulent flow; the relations (15a) to (18) reduce to:

$$\bar{w} = -\frac{a_0}{m^2 \varepsilon} \frac{\partial p}{\partial z} (R^m - r^m) \tag{21_1}$$

$$\bar{Q}_0 = \frac{m}{(m+2)} \frac{a_0 \pi}{m^2 \varepsilon} \frac{\partial p}{\partial z} R^{m+2} \qquad (m \geqq 2) \tag{21_2}$$

$$\frac{\bar{w}}{\bar{c}} = \frac{m+2}{m} \left(1 - \frac{r^m}{R^m}\right) \tag{21_3}$$

In this case we should not forget - as we have already emphasized this - that a_0 also could be a function of R; that is,

$$a_0 = a_1 / R^{\alpha}. \tag{22}$$

From (21_3) it follows for $r = 0$ immediately that

$$\bar{w}_0 = \frac{m+2}{m} \bar{c} \qquad (m \geqq 2) \tag{21_4}$$

and in this case m is always to be chosen so that (21_4) agrees with (20); however, we must not forget that the relation (21_1) also must agree with the results of measurements. Let us make the following table therefore, where we shall consider only some integral values of m:

m	2	3	4	5	6	7	8	9	10
(m + 2)/m :	2	1.667	1.5	1.4	1.333	1.286	1.25	1.222	1.2

m	11	12	13	100	5000	∞
(m + 2)/m	1.182	1.167	1.154	1.02	1.000	1

Only for the case m = 2 is it valid for laminar flow, since then we have n = m -2 = 0; all other cases m > 2 represent turbulent flows for us. In Figure 1 the new laws for various values of m are represented graphically. We will now assign these values numerically:

r/R	$\bar{w} / \bar{c}$ m = 2	m = 3	m = 6	m = 12	m = 100	m = ∞
0.000	2.000	1.667	1.333	1.167	1.020	1.000
0.125	1.969	1.664	1.333	1.167	1.020	1.000
0.250	1.875	1.641	1.333	1.167	1.020	1.000
0.375	1.789	1.610	1.331	1.167	1.020	1.000
0.500	1.500	1.459	1.312	1.167	1.020	1.000
0.625	1.219	1.260	1.254	1.163	1.020	1.000
0.750	0.875	0.964	1.096	1.130	1.020	1.000
0.875	0.469	0.550	0.735	0.921	1.020	1.000
0.937	0.244	0.296	0.431	0.633	1.018	1.000
0.999	0.004	0.005	0.008	0.014	0.096	1.000
1.000	0.000	0.000	0.000	0.000	0.000	0.000

Immediately apparent from the figure is the familar fact that during turbulent flow the velocity is distributed [9] much more uniformly over the cross-section than during laminar flow. In this case, however, we have not obtained complete agreement with the measurements, especially not in the immediate neighborhood of the pipe's wall.

3. The most general case: Here we have four quantities A, a, b, m at our disposal, so that we can fit very exactly the theory to observations. In order to show this, we shall now compute an example. First we must determine the number m, and indeed from the relation (18) we set r = R and

$$b = ga; \tag{23}$$

it follows immediately that:

$$g = \frac{R^{m-2} \cdot \left[\frac{\bar{w}_0/\bar{c}}{m(m+2)} - \frac{1}{m^2}\right]}{\frac{1}{4} - \frac{\bar{w}_0/\bar{c}}{8}} . \tag{24}$$

In this way we can eliminate in (18) the unknown quantity a and finally obtain:

$$\frac{\bar{w}}{\bar{c}} = \frac{\left[1 - \frac{r^m}{R^m}\right] + \frac{\frac{m}{m+2} \cdot \frac{\bar{w}_0}{\bar{c}} - 1}{1 - \frac{1}{2}\frac{\bar{w}_0}{\bar{c}}} \cdot \left[1 - \frac{r^2}{R^2}\right]}{\frac{m}{m+2} + \frac{\frac{m}{m+2}\frac{\bar{w}_0}{\bar{c}} - 1}{2\left(1 - \frac{1}{2}\frac{\bar{w}_0}{\bar{c}}\right)}} . \tag{25}$$

This formula is very convenient for practical computation. For a carefully smoothed cement pipe of R = 40 cm radius, Bazin has found [10] $\bar{w}_0 / \bar{c} = 1.167$ and in the "immediate neighborhood" of the pipe's wall $\bar{w} / \bar{c} = 0.741$. Since, according to the investigations of Forchheimer [11], a value r = 39.96 cm lies closer for the last-named case, it follows that we have for m a value between 2000 and 3000. It is very interesting here that the number m is not too very "sensitive"; this number, however, cannot exceed in our case the value 5000. We shall now compare the results of Bazin's measurements with the results of our theory:

r/R	$\overline{w}$ / $\overline{c}$ (obs.)	m = 2200 $\overline{w}$ / $\overline{c}$ (calc.)	m = 3000 $\overline{w}$ / $\overline{c}$ (calc.)
0.000	1.167	1.167	1.167
0.125	1.160	1.162	1.162
0.250	1.147	1.146	1.146
0.375	1.126	1.120	1.120
0.500	1.092	1.084	1.084
0.625	1.047	1.037	1.037
0.750	1.001	0.980	0.980
0.875	0.922	0.912	0.912
0.937	0.846	0.875	0.875
0.999	0.741	0.741	0.792
1.000	0.000	0.000	0.000

The agreement is really an excellent one [12]! In the case of the value measured by Bazin 0.741 is valid for still larger neighborhoods than 0.04 cm at the pipe's wall; thus we shall keep the value m = 3000. In this case we must keep in mind that we have maintained here at all times the "virtual" internal friction or "turbulence" ε constant throughout; nevertheless our theory governs perfectly the turbulent flow considered.

After we have determined the number m, we can determine also g by means of relation (24), and the formula (16) will assume the following form:

$$\overline{Q}_0 = \frac{A a}{\varepsilon} \cdot \frac{\partial p}{\partial z} \cdot \left[\frac{m}{m+2} \cdot \frac{\pi}{m^2} + \frac{\pi}{2} \cdot \frac{\left(\frac{\overline{w}_0/\overline{c}}{m(m+2)} - \frac{1}{m^2} \right)}{\left(1 - \frac{1}{2} \frac{\overline{w}_0}{\overline{c}} \right)} \right] \cdot R^{m+2} \,, \qquad (26)$$

where we must still insert:

$$A = 1, \tag{27$_1$}$$

$$a = \cancel{K}/R^{\beta} \qquad a = k/R^{\beta} \quad [\text{small "k"}] \tag{27$_2$}$$

Now it is always possible, on the basis of measurements, to determine the
k constants K and β. Moreover it is completely sufficient if we determine
k/ε only the ratio K/ε, since only $\bar{\theta}_3/\varepsilon$ appears throughout at all times [14]. Our problem is therefore completely solved for the case where the average velocities are stationary [15].

Now let us return to the formulas (23) and (24) and let us ask ourselves in which case we shall have: $b = 0$. This will be possible only if in (24) the numerator is equal to zero; that is, we then obtain the condition (21_4) already known.

II. We shall now let the condition fall, namely that the average velocities are stationary; that is, we shall consider the more general case where we have: $\partial\bar{w} / \partial t \neq 0$. Our basic equations ($4_{1,2}$) will be reduced to:

$$\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial y^2} = \frac{\bar{\theta}_3}{\varepsilon}\frac{\partial p}{\partial z} - \frac{\rho}{\varepsilon}\frac{\partial \bar{w}}{\partial t} \tag{30}$$

or, for the sake of axial symmetry:

$$\frac{\partial}{\partial r}\left(r\frac{\partial \bar{w}}{\partial r}\right) = \left(\frac{\bar{\theta}_3}{\varepsilon}\frac{\partial p}{\partial z} - \frac{\rho}{\varepsilon}\frac{\partial \bar{w}}{\partial t}\right) r \tag{30a}$$

Let us set similarly as before:

$$\bar{\theta}_3 = A\cdot\left(b + a r^{m-2}\right) + \Phi(r,t), \tag{31}$$

where A is a function of time; we then obtain:

$$\frac{\partial}{\partial r}\left(r\frac{\partial \bar{w}}{\partial r}\right) = \frac{A}{\varepsilon}\frac{\partial p}{\partial z}\left(b + a r^{m-2}\right) + \frac{1}{\varepsilon}\frac{\partial p}{\partial z}\Phi(r,t) - \frac{\rho}{\varepsilon}\frac{\partial \bar{w}}{\partial t}. \qquad (32)$$

Obviously it would not be easy to solve generally this differential equation; let us therefore consider an especially simple case where the following expression holds:

$$\Phi(r,t) = \left(\rho : \frac{\partial p}{\partial z}\right)\cdot\frac{\partial \bar{w}}{\partial t} \quad ; \qquad (33)$$

the equation (32) reduces then to the equation already known to us:

$$\frac{\partial}{\partial r}\left(r\frac{\partial \bar{w}}{\partial r}\right) = \frac{A}{\varepsilon}\frac{\partial p}{\partial z}\left(b + a r^{m-2}\right)\cdot r \quad ; \qquad (34)$$

this equation we have already solved earlier according to the methods (15), or (15a). Let us go still a step further: Let us designate by ℓ the length of the pipe and by π

$$\pi = p_1 - p_2 \qquad (35)$$

the difference in pressure at the front and end of the cylinder, and if we set

$$\bar{A}_T = \frac{\varepsilon \bar{Q} \ell}{\frac{m}{m+2}\cdot\frac{a\pi}{m^2}R^{m+2} + \frac{b\pi}{8}R^4} \qquad (36)$$

where $\bar{Q}$ means the flow, not per second, but in the time t, then we obtain from (16), by multiplying the right side by t, the following expression:

$$\pi = \frac{\bar{A}_T}{A}\cdot\frac{1}{t} \qquad (37)$$

We must now still determine the function A of t on the basis of measurements. The measurement of O. Reynolds, Couette and E. Bose and co-workers yield for turbulent flows the following law:

$$\Pi = \bar{A}_T \cdot \left(\frac{1}{t}\right)^s \tag{38}$$

where $\bar{A}_T$ is a constant and s equals 1.6 ~ 1.95 [16]. Since in our law (36) $\bar{A}_T$ also is a constant, then we have

$$A = t^{s-1}; \tag{39}$$

and therefore we have "deduced" the empirical law (38) also theoretically [17]. We see, however, that the empirical law (38) is only an entirely special case of our theory; or law (37) is of a much more general nature. It must be still emphasized expressly that the formulas (15a) to (18) have also here their validity; only, the quantity A is now a function of the time t. The relation (18) is now independent of the time t.

Since the quantity $\bar{A}_T$ is experimentally determinable, we can use its value in order to determine more closely the quantity a from (36). Thus, for example, von Karman [18] has shown that $\bar{A}_T$ is also a function of the thickness ρ; accordingly it results from (36) that the quantity a can likewise be a function of ρ. In other words, it is possible to determine entirely [see I, (27_2) here] the quantities a and b = ga on the basis of measurements. Further we have found that

$$\bar{\Theta}_3 = A(b + ar^{m-2}) + \left(\rho : \frac{\partial p}{\partial z}\right) \cdot \frac{\partial \bar{w}}{\partial t} \; ; \tag{40}$$

since $\bar{\Theta}_3$ is a function of the average velocity $\bar{w}$, we understand why turbulence must appear if the average velocity $\bar{w}$ exceeds a certain limit. Further investigations I have reserved for a later larger publication; I wanted here to indicate only the range and importance of my expansion of the theory, of which I hope that it can perform more than the ingenious theory of Boussinesq, since it can yield us by means of equations ($5_{1,2}$) also the information on pulsations [19].

In conclusion I note that I had found cause to expand my theory published in the year 1920, because I was invited by the Committee in the Section for Hydro- and Aerodynamics of the International Congress for Applied Mechanics at Delft (Holland) to give a report on my investigations on the turbulence of winds [20]. Further investigations will yet show how far the method started here will prove useful.

Conclusion

First, new basic equations are presented for the turbulent motion of a fluid with internal friction. After the author has shown already earlier such an expansion of the theory in his investigation of the structure of winds, he considers here turbulent flow in round pipes. In this case it is shown that one can bring the theory into almost complete agreement with observations. The theory also offers transitional flows between the laminar and turbulent state.

(Submitted 15 July 1924)

Notes

1) See, for example, Ph. Forchheimer, Hydraulik. Leipzig and Berlin: 1914, page 26. R. von Mises, Elements of Engineering Hydromechanics. Part I. Leipzig and Berlin: 1914, pages 34-5. Cl. Schaefer, Introduction to Theoretical Physics. Volume I, 2. A. Berlin and Leipzig: 1922, pages 909-911.

2) See, for example, J. Boussinesq, Theory of the Turbulent and Tumultuous Flow of Fluids. Paris: 1897. See also Ph. Forchheimer, loco citato, pages 26 et seq., 114 et seq.

3) See, for example, H. Lamb, Textbook of Hydrodynamics (German translation by J. Friedel). Leipzig and Berlin: 1907, pages 735 et seq. (especially pages 743 et seq.).

4) S. Mohorovičić, Investigations of the Wind in Radziechow, in Galicia. (Part I: Measurements. Part II: Manipulation of the Measurements and Theoretical Considerations on the Structure of Winds with Special Reference to Turbulence). Compare the results with the measurements. Bulletin de l'Academie des sciences et des Arts de Slaves du Sud de Zagreb (Croatie) Volume 13-14: pages 85-125; Zagreb: 1920. (see also the author's abstract in the Physikalische Berichte /Physical Reports/, II, 183, 1921).

5) See, for example, F. M. Exner, Dynamic Meteorology. Leipzig and Berlin: 1917, pages 107. Also see Ph. Forchheimer, loco citato, page 27.

7) S. Mohorovičić, loco citato, page 113.

6) See, for example, H. Lamb, loco citato, page 744

8) See R. von Mises, loco citato, page 66.

9) See, for example, R von Mises, loco citato, page 66. Also see Ph. Forchheimer, loco citato, page 114.

10) See, for example, R. Von Mises, loco citato, pages 66-67. In this series of tests the average mean-speed $\bar{c}$ was about 110 ~ 150 cm/sec.

11) Loco citato, page 120.

12) I must note here that an entirely different solution of this problem was given by Th. von Karman ("On Laminar and Turbulent Friction", Zeitschrift fur Applied Matematik und Mechanik I (1921), 233.) Also see Abhandlungen aus dem aerodynamischen Institut an der technischen Hochschule Aachen, Lif. 1; Aachen: 1921.

13) It is namely: $\bar{\theta}_s/\varepsilon = \frac{k}{\varepsilon R}\beta\cdot(g + r^{m-2})$. (28)

14) Here we keep ε constant for the following reason: the measurements of Ludwig Schiller ("Roughness and Critical Numbers: An experimental Report on the Problem of Turbulence". Zeitschrift fur Physik 3 (1920), 412) have shown that the roughness of the pipe's wall is not a necessary condition for the genesis of turbulence, but if turbulence arises in one place then it will continue with the flow.

15) We could have also inserted the series:

$$\bar{\Theta}_3 = b + a_1 r + a_2 r^2 + \dots + a_n r^n, \qquad (29)$$

and then solved the problem; however, this would have been an unnecessary complication here. Perhaps such cases also are possible where we must employ the more exact form (29) for $\bar{\Theta}_3$.

16) Cl. Schaeffer, loco citato, page 906. Bose found s = 1.620 (see page 911). On this occasion I must call attention to the extremely valuable investigation of von Karman, Physikalische Zeitschrift 12 (1911), 283.

17) Cl. Schaefer says in his familiar textbook, loco citato, page 911, "On the other hand the efforts of the most prominent theoretician (Reynolds, Lorentz, Sommerfeld, Hamel, Boussinesq) have not succeeded up to now in delivering a satisfactory theory of turbulent flow, whose problem it should be above all things to derive on the basis of a clear knowledge of the actual flow process the experimentally found equation of turbulence (129) /in our case, equation (38) here/. We must consider as one of the most important aims of hydrodynamics the solution of this problem".

18) Loco citato, page 284. See also Cl. Schaefer, loco citato, paged 910-911.

19) I have not accomplished this here, since the paths of individual fluid particles do not interest us here (only being preliminary). It is just this, however, that is the most important problem in aerodynamics. Turbulent wind flows play an enormous role in aeromautics. And it was just this fact that gave me the first impulse, on the one hand, to attack this problem and especially to treat pulsations theoretically (see my work cited above).

20) Professor J. M. Burgers (Delft, Holland) has compiled an abstract of this work for the proceedings of the Congress, for which I express to him my deepest gratitude.

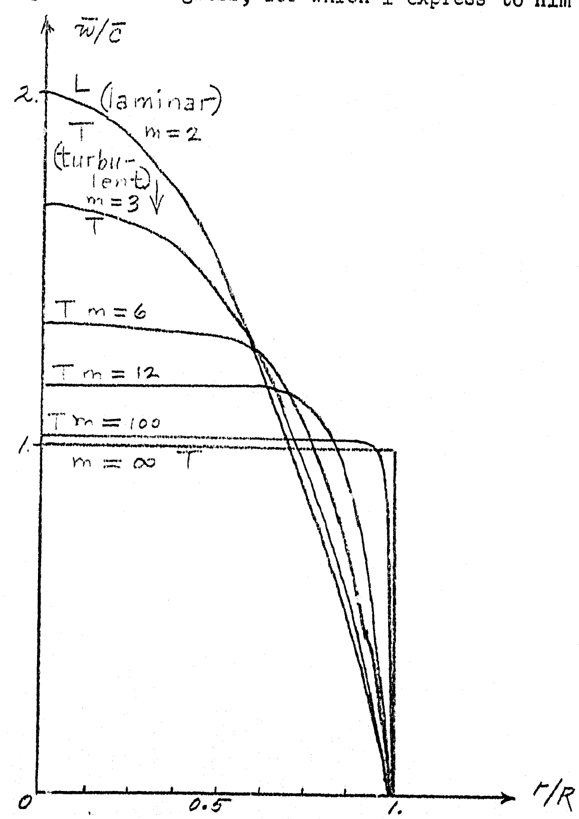

Figure 1. Distribution law of velocities at various distances from the axis of a round pipe, for laminar (L) and turbulent (T) flow. (m is the "degree" of the turbulent state.)

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