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ON THE STRENGTH OF THIN SHELLS UNDER FINITE DEFORMATION. 1. FUNDAMENTAL THEORY

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ABSTRACT

The general theory of thin shells was formerly investigated by A. E. H. Love,[1] and has been applied to many problems of stability of thin shells and their results of calculation compared with experiments by various investigators. Generally it was clear that some regular discrepancies existed between results deduced from Love's theory and experimental evidences.

Th. von Karman pointed out that Love's theory based on infinitely small deformation of shells is not realized in many cases; actually deformations of shells by external or end pressure are so called "Durchschlag" /Note: The provisional Japanese translation used by the author for this word is <u>tenkutsu</u>, which literally means "rebending" according to the separate Chinese characters;

whereas "Durchschlag" means "punch through".7, and consequently the stability of thin shells must be discussed taking into account finite deformation based on nonlinear differential equations of equilibrium. But his theory has not yet been completed.

The author first investigates the general theory of elasticity taking into account finite displacement; then he obtains the general theory of thin shells for the case of large deformations, which forms the generalization of Love's thin-shell theory; and lastly, as its special case, he shows the fundamental equation of "Durchschlag" by the method of derivation of equations referred to some shell shapes.

I. INTRODUCTION

The general theory of deformations in shells /literally, "thin curved plates"/ was first discussed by A. E. H. Love [1] on the basis of the assumption that these deformations are infinitely small, and later many other investigators carried out calculations in application to the problem of the buckling of shells in order to make comparisons with experimental results; the general conclusion, however, was that the buckling values of a shell which possesses initial curvature is several times larger than the experimental value. This was explained by Th. von Karman as due to "Durchschlag". /Note: The Japanese expression, tenkutsu, employed by the author here to translate "Durchschlag" /"break through"/ means literally "rebending". It is interesting to note that the Japanese word, zakutsu, for "buckling" means literally "collapse-bending"./ that is, the early problem of buckling was a theory employing the assumption of infinitely small deformations, and hence its buckling values were sought as the eigenvalues of a linear differential equation. Concerning the phenomenon of deformations in shells possessing initial curvature, if one takes into consideration finite deformation in shells, it is apparent that there exists an equilibrial state which can maintain equilibrium by an external pressure lower than the above-mentioned buckling value; before the external pressure reaches the buckling value, the shell leaves the state of infinitely small deformation assumed by Love and thus discontinuously experiences finite deformations. The latter is often regarded as the buckling load in experiments. This represents the so-called phenomenon of Durchschlag and, if finite deformation is not taken into consideration, cannot be interpreted.

The theory of shells taking into consideration finite deformation has been studied in connection with particular shells by Karman [2,3], KAWANO [4], MIZUKI [5], YOTSUYA [6]; but, as even Karman himself has stated, the theory of Durchschlag has not yet been perfected. Concerning phenomenon besides Durchschlag, often one must consider states of finite deformation in order to decide the ultimate strength of shells if extremely thin. Since, however, there has been no literature, as far as the author knows, giving the general theory of finite deformed states, the author has attempted here to treat this subject.

In studying the finite deformations of shells one must first review the general theory of elastic bodies subjected to finite deformation, which theory is basic to the discussion here. Several studies [7,8,9] on this have already appeared, but since they are thought unsuitable as a foundation for any theory of deformation of shells, the author will first endeavor to obtain the equations of equilibrium relative to a coordinate system which deforms (transforms) together with the elastic body; next he will apply these equations to the deformation of shells to obtain their fundamental equations. Then, after carrying out some studies in connection with Durchschlag, the author will derive the general equations and finally will show a method for obtaining definite concrete equations for several special cases /rectangular and polar coordinates/.

II. FINITE DISTORTION (STRAIN) OF AN ELASTIC BODY

In Figure 1 take any small particle $\mathfrak{P}$ in the elastic body; to indicate the position of $\mathfrak{P}$ express by any curvilinear coordinate system $R_1(\xi^1, \xi^2, \xi^3)$ which deforms (transforms) along with the body, and by a rectangular coordinate system $R_0(x^1, x^2, x^3)$, which is held fixed in space. (Note: The numbers in x^1, x^2, x^3 are not exponents but express mere indices. Indices assume the values 1, 2, 3. In what follows we shall use this notation.) Take the position of $\mathfrak{P}$ before deformation as P, and the position after deformation as 'P. Taking R_1 before deformation, we find the relation between (x^1, x^2, x^3) and (ξ^1, ξ^2, ξ^3) to be

$$x^i = f^i(\xi^1, \xi^2, \xi^3) \tag{2.1}$$

Next, take another particle $\mathfrak{Q}$ which is very close to $\mathfrak{P}$; take the position of $\mathfrak{Q}$ before deformation as Q, and the position after deformation as 'Q. The coordinates R_0 and R_1 of $\mathfrak{Q}$ are respectively $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ and $(\xi^1 + d\xi^1, \xi^2 + d\xi^2, \xi^3 + d\xi^3)$; if the distance between $\mathfrak{P}$ and $\mathfrak{Q}$ before deformation is taken as dr, then dr is given by the following equation

$$dr^2 = \delta_{ij}\, dx^i dx^j \qquad \left\{ \delta_{ij} = \begin{matrix} 1 & (i = j) \\ 0 & (i \neq j) \end{matrix} \right\} \tag{2.2}$$

Next, if we substitute into (2.2) the expression obtained from (2.1), namely,

$$dx^i = \frac{\partial f^i}{\partial \xi^\mu} d\xi^\mu$$

we obtain

$$dr^2 = \delta_{ij} \frac{\partial f^i}{\partial \xi^\mu} \frac{\partial f^j}{\partial \xi^\nu} d\xi^\mu d\xi^\nu \equiv g_{\mu\nu} d\xi^\mu d\xi^\nu \tag{2.3}$$

where we take

$$g_{\mu\nu} = \delta_{ij} \frac{\partial f^i}{\partial \xi^\mu} \frac{\partial f^j}{\partial \xi^\nu} .$$

(Note: Hereafter, the Roman letters i, j are used to designate the various quantities relating to R_o; and the Greek letters μ, ν are used to designate the various quantities relating to R_1. When the same index appears twice in the same term, we shall agree to the usual tensor summation convention - that is, in every such case we sum from 1 to 3 relative to the repeated index.)

Next let the elastic body suffer a deformation. If the coordinates of relative to R_o are taken as $^1x^i = x^i + \mu^i(x)$, the $\mu^i(x)$ represent the components, relative to R_o, of the displacement vector. When we have expressed $^1x^i$ and μ^i as functions of ξ^μ, and if we assume

$$^1x = \varphi^i(\xi), \qquad \mu^i = \nu^i(\xi)$$

then the R_o coordinates of ξ^μ after a deformation become

$$\varphi^i(\xi) = f^i(\xi) + \nu^i(\xi) . \tag{2.4}$$

Consequently the R_0 coordinates of $\mathfrak{Q}$ after a deformation become

$$'x^i + d'x^i = \varphi^i(\xi) + d\varphi^i(\xi) = f^i(\xi) + df^i(\xi) + v^i(\xi) + dv^i(\xi);$$

therefore the distance d'r between $\mathfrak{P}^{\xi}$ and $\mathfrak{Q}$ after a deformation is expressed as follows:

$$\begin{aligned}
d'r^2 &= \delta_{ij}\, d'x^i d'x^j = \delta_{ij} \frac{\partial\varphi^i}{\partial\xi^\mu}\frac{\partial\varphi^j}{\partial\xi^\nu} d\xi^\mu d\xi^\nu \\
&= \delta_{ij}\left(\frac{\partial f^i}{\partial\xi^\mu} + \frac{\partial v^i}{\partial\xi^\mu}\right)\left(\frac{\partial f^j}{\partial\xi^\nu} + \frac{\partial v^j}{\partial\xi^\nu}\right) d\xi^\mu d\xi^\nu \\
&= \left(g_{\mu\nu} + \delta_{ij}\frac{\partial f^i}{\partial\xi^\mu}\frac{\partial v^j}{\partial\xi^\nu} + \delta_{ij}\frac{\partial f^j}{\partial\xi^\nu}\frac{\partial v^i}{\partial\xi^\mu} + \delta_{ij}\frac{\partial v^i}{\partial\xi^\mu}\frac{\partial v^j}{\partial\xi^\nu}\right) d\xi^\mu d\xi^\nu \\
&\equiv {'g_{\mu\nu}}\, d\xi^\mu d\xi^\nu .
\end{aligned}$$

Again, if we set $d'r^2 = dr^2 + 2\varepsilon_{\mu\nu} d\xi^\mu d\xi^\nu$,

we get

$$\begin{aligned}
\varepsilon_{\mu\nu} &= \frac{1}{2}\left({'g_{\mu\nu}} - g_{\mu\nu}\right) \\
&= \frac{1}{2}(\delta_{ij})\left(\frac{\partial f^i}{\partial\xi^\mu}\frac{\partial v^j}{\partial\xi^\nu} + \frac{\partial f^j}{\partial\xi^\nu}\frac{\partial v^i}{\partial\xi^\mu} + \frac{\partial v^i}{\partial\xi^\mu}\frac{\partial v^j}{\partial\xi^\nu}\right). \qquad (2.5)
\end{aligned}$$

Further, if we assume, relative to another coordinate system $\bar{R}_1(\bar{\xi}^1, \bar{\xi}^2, \bar{\xi}^3)$ which varies with the body, the expression

$$d'r^2 - dr^2 = 2\bar{\varepsilon}_{\lambda\kappa} d\bar{\xi}^\lambda d\bar{\xi}^\kappa ,$$

we clearly can establish that

$$\bar{\varepsilon}_{\lambda\kappa} = \frac{\partial\xi^\mu}{\partial\bar{\xi}^\lambda} \cdot \frac{\partial\xi^\nu}{\partial\bar{\xi}^\kappa}\, \varepsilon_{\mu\nu} ;$$

therefore, we find that $\varepsilon_{\mu\nu}$ are the components of a second-order covariant tensor relative to R_1 and moreover are symmetric according to (2.5).

Next take the components of the displacement vector relative to R_1 as w^λ, if using its covariant derivative $w^\lambda{}_{,\mu}$, that is

$$w^\lambda{}_{,\mu} \equiv \frac{\partial w^\lambda}{\partial \xi^\mu} + \Gamma^\lambda_{\mu\nu} w^\nu \qquad \left(\Gamma^\lambda_{\mu\nu} = g^{\lambda\alpha}\left[\frac{\partial g_{\alpha\mu}}{\partial \xi^\nu} + \frac{\partial g_{\alpha\nu}}{\partial \xi^\mu} - \frac{\partial g_{\mu\nu}}{\partial \xi^\alpha}\right] \right)^*$$

we transform v^1 in the right side of (2.5) into w^λ we get

$$\varepsilon_{\mu\nu} = \frac{1}{2}\left(w_{\nu,\mu} + w_{\mu,\nu} + g_{\kappa\lambda} w^\lambda{}_{,\mu} w^\kappa{}_{,\nu}\right). \tag{2.6}$$

(Note: We call provisionally the tensor $\varepsilon_{\mu\nu}$ the distortion tensor. See McConnell [10]). The third term on the right side of (2.6) is the term that expresses anew the fact that we are considering finite deformations.

The variation which arises due to distortion, of the coordinate system of the Christoffel symbol $\Gamma^\lambda_{\mu\nu}$, relative to the coordinate system, which deforms together with the elastic body, includes the displacement and its derivative when appearing in the equilibrium equations; hitherto it had been disregarded, but in the theory of finite deformation it is necessary to take it too into consideration. Consequently if, taking the Christoffel symbol after deformation as $'\Gamma^\lambda_{\mu\nu}$ we take up to second-order terms of $\varepsilon_{\mu\nu}$ we get

$$'\Gamma^\lambda_{\mu\nu} \doteqdot \Gamma^\lambda_{\mu\nu} + g^{\lambda\alpha}\left(\frac{\partial \varepsilon_{\alpha\mu}}{\partial \xi^\nu} + \frac{\partial \varepsilon_{\alpha\nu}}{\partial \xi^\mu} - \frac{\partial \varepsilon_{\mu\nu}}{\partial \xi^\alpha}\right) - \gamma^{\lambda\alpha}\left(\frac{\partial g_{\alpha\mu}}{\partial \xi^\nu} + \frac{\partial g_{\alpha\nu}}{\partial \xi^\mu} - \frac{\partial g_{\mu\nu}}{\partial \xi^\alpha}\right), \tag{2.7}$$

where

$$'g^{\mu\nu} = g^{\mu\nu} - 2\gamma^{\mu\nu}$$
$$\gamma^{\mu\nu} = \sqrt{g^{(\mu\mu)} g^{(\nu\nu)}}$$

(we do not take the sum relative to $(\mu\mu)$, $(\nu\nu)$).

*[Note: Sic. The usual coefficient $\frac{1}{2}$ was omitted in the original.]

Tensor $\varepsilon_{\mu\nu}$ is a quantity concerned with distortion, but is not the distortion itself. Now if we take $\eta_{(\mu)}$ as the component of elongation relative to coordinate ξ^{μ} then we get

$$\eta_{(\mu)} = \sqrt{\frac{'g_{\mu\mu}}{g_{\mu\mu}}} = \sqrt{1+\overset{\circ}{\varepsilon}_{\mu\mu}} - 1 = \overset{\circ}{\varepsilon}_{\mu\mu} - \frac{1}{2}\cdot\frac{1}{4}(2\overset{\circ}{\varepsilon}_{\mu\mu})^2 + \frac{1}{2}\cdot\frac{1\cdot 3}{4\cdot 6}(2\overset{\circ}{\varepsilon}_{\mu\mu})^3 - \ldots$$

(the sum relative to μ is not taken)

where $\overset{\circ}{\varepsilon}_{\mu\mu} = \frac{\varepsilon_{\mu\mu}}{g_{\mu\mu}}$ expresses the physical components of $\varepsilon_{\mu\mu}$. When terms in $\overset{\circ}{\varepsilon}_{\mu\mu}$ of degree higher than the second can be omitted, our work is in agreement with previous theories. If we take $\zeta_{(\mu\nu)}$ as the angle formed by the ξ^{μ}- curve and ξ^{ν}- curve before deformation, we get

$$\cos\zeta_{(\mu\nu)} = \frac{2\varepsilon_{\mu\nu}}{\sqrt{g_{\mu\mu}+2\varepsilon_{\mu\mu}}\sqrt{g_{\nu\nu}+2\varepsilon_{\nu\nu}}} = \frac{2\overset{\circ}{\varepsilon}_{\mu\nu}}{\sqrt{1+2\overset{\circ}{\varepsilon}_{\mu\mu}}\sqrt{1+2\overset{\circ}{\varepsilon}_{\nu\nu}}}$$

(the usual summation convention does not apply here to (μ, ν).)

When we can omit terms in $\overset{\circ}{\varepsilon}_{\mu\nu}$ above the second degree, we get

$$\zeta_{(\mu\nu)} = \frac{\pi}{2} + \overset{\circ}{\varepsilon}_{\mu\nu}$$

which is in agreement with previous theories.

Again, if we seek suitable conditions, we get

$$\varepsilon_{\mu\nu,\sigma\tau} + \varepsilon_{\sigma\tau,\mu\nu} - 2g^{\lambda\kappa}w_{\lambda,\mu\nu}w_{\kappa,\sigma\tau} = \varepsilon_{\mu\sigma,\nu\tau} + \varepsilon_{\nu\tau,\mu\sigma} - 2g^{\lambda\kappa}w_{\lambda,\nu\tau}w_{\kappa,\mu\sigma} \quad (2.8)$$

the third terms on both sides are newly added terms.

III. LAW OF ELASTICITY AND EQUATIONS OF MOTION

Take S as the area of any closed curve in an elastic body after deformation, and take V as the volume of the part surrounded by S. If we designate by T the kinetic energy possessed by V, and designate by ρ the density after deformation, we have

$$T = \frac{1}{2}\int_V \rho\, g_{\mu\nu}\dot{\xi}^{\mu}\dot{\xi}^{\nu}\, dV \qquad (3.1)$$

(Note: ξ^{μ} is the coordinate that deforms along with the solid.) The dot ($\cdot$) designates differentiation with respect to the time t, and we use the designation $dV = \sqrt{g}\cdot d\xi^1 d\xi^2 d\xi^3$. If we take K^{μ} as the components of the resultant stress vector operating on V, and M^{μ} as the components of the momentum vector, and if we designate by δW the virtual work done by K^{μ} and M^{μ} we then have

$$\delta W = \int_S K^{\mu}\delta\xi_{\mu}\, dS + \int_V \rho M^{\mu}\delta\xi_{\mu}\, dV. \qquad (3.2)$$

where

$$dS_{\lambda} = \sqrt{g}\cdot e_{\lambda\mu\nu}\, d\xi^{\mu} d\xi^{\nu}$$

$$e_{\lambda\mu\nu} = \begin{cases} 0 & \text{(when two of } \lambda,\mu,\nu \text{ are equal)} \\ 1 & \text{(when } \lambda,\mu,\nu \text{ are in even order 1,2,3)} \\ -1 & \text{(when } \lambda,\mu,\nu \text{ are in odd order 1,2,3)} \end{cases}$$

If we take b^{μ} as the components of the velocity of [illegible], we get

$$b^{\mu} = \frac{d^2\xi^{\mu}}{dt^2} + \Gamma^{\mu}_{\alpha\beta}\frac{d\xi^{\alpha}}{dt}\cdot\frac{d\xi^{\beta}}{dt};$$

further, for any two times t_0 and t_1, we have

$$\int_{t_0}^{t_1} \delta T\, dt = -\int_{t_0}^{t_1}\int_V \rho\, b^{\mu}\delta\xi_{\mu}\, dV. \qquad (3.3)$$

Again, introducing the stress tensor $T^{\mu\nu}$ by means of $K^{\mu}dS = T^{\mu\nu}dS_{\nu}$ and employing Gauss' divergence theorem, we get

$$\int_S (K^{\mu}dS)\,d\xi_{\mu} = \int_V \left(T^{\mu\nu}{}_{,\nu}\, d\xi_{\mu} + T^{\mu\nu}(\delta\xi_{\mu})_{,\nu}\right) dV. \qquad (3.4)$$

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Hamilton's principle is

$$\delta \int_{t_0}^{t} (T + W)\, dt = 0 \; ; \tag{3.5}$$

therefore, if we substitute δW obtained by substitution of (3.4) into (3.2), and (3.3) into (3.5), we obtain

$$\int_0^{t_1} \int_V \{ (T^{\mu\nu}{}_{;\nu} + {}^{\prime}\!\rho M^{\mu} - {}^{\prime}\!\rho b^{\mu}) \delta\xi_{\mu} + T^{\mu\nu}(\delta\xi_{\mu})_{;\nu} \}\, dV dt = 0 . \tag{3.6}$$

In the above formula, $\delta\xi_{\mu} = g_{\mu\nu}\delta\xi^{\nu}$ is any arbitrary virtual displacement; therefore, if a virtual displacement $\partial(d'r^2) = 0$, consequently $\delta(g_{\mu\nu} d\xi^{\mu} d\xi^{\nu}) = 0$, is taken, we have $\delta\xi_{\mu\nu} = 0$ for all $d\xi^{\mu}$ - that is, no distortion is produced relative to such virtual displacements. Also, since we have

$$\delta(d'r^2) = \delta({}^{\prime}g_{\mu\nu} d\xi^{\mu} d\xi^{\nu}) = \{ (\delta\xi_{\mu})_{,\nu} + (\delta\xi_{\nu})_{,\mu} \} d\xi^{\mu} d\xi^{\nu} = 0 ,$$

the last term in (3.6) becomes equal to zero 0. From the fact that (3.6) should hold for arbitrary t_1 and V, we get the equation of motion

$$T^{\mu\nu}{}_{;\nu} + {}^{\prime}\!\rho M^{\mu} - {}^{\prime}\!\rho b^{\mu} = 0 . \tag{3.7}$$

Each term in the above equation expresses the components of the various quantities relative to the coordinate system after deformation.

Also, the virtual work at each time becomes finally

$$\delta W + \delta T = \int_V T^{\mu\nu}(\delta\xi_{\mu})_{,\nu}\, dV \; ; \tag{3.8}$$

furthermore, since $\delta W + \delta T = 0$, we get $T^{\mu\nu} = T^{\nu\mu}$.

If we designate by ${}^{\prime}\!\rho\phi$ the elastic potential, we get from (3.8)

$${}^{\prime}\!\rho\delta\phi = T^{\mu\nu}(\delta\xi_{\mu})_{,\nu} = T^{\mu\nu}\delta\varepsilon_{\mu\nu} . \tag{3.9}$$

Since ϕ is a function of $\varepsilon_{\mu\nu}$, we have

$$\delta\phi = \frac{\partial\phi}{\partial\varepsilon_{\mu\nu}}\delta\varepsilon_{\mu\nu} \; ; \tag{3.10}$$

comparing (3.9) and (3.10) we get

$$T^{\mu\nu} = {}'\rho\frac{\partial\phi}{\partial\varepsilon_{\mu\nu}} \tag{3.11}$$

Since ϕ is an invariant, we must obtain $\varepsilon_{\mu\nu}$ in the form of $I_1 = \varepsilon^{\mu}_{\mu}$, $I_2 = \frac{1}{2!}\delta^{\nu_1\nu_2}_{\mu_1\mu_2}\,\varepsilon^{\nu_1\nu_2}_{\mu_1\mu_2}$, $I_3 = \det \varepsilon^{\nu}_{\mu}$ $^{9)}$. Furthermore, $'\rho$ also is a function of $\varepsilon_{\mu\nu}$; the ratio with the density ρ before deformation is given by

$$\begin{aligned} {}'\rho/\rho &= \det {}'g_{\mu\nu}/\det g_{\mu\nu} \\ &= \sqrt{1+2I_1+3I_2+8I_3}\;. \end{aligned}$$

Consequently, now taking λ and μ as constants (Note: There are no relations among the indices λ, μ, ν), and setting

$$\rho\phi = \tfrac{1}{2}(\lambda+2\mu)I_1^2 - 2\mu I_2 \tag{3.12}$$

we get

$$T_{\sigma\tau} = \sqrt{1+2I_1+3I_2+8I_3}\cdot\left(\lambda I_1 {}'g_{\sigma\tau} + 2\mu\varepsilon_{\sigma\tau}\right). \tag{3.13}$$

In finite distortions, when we can disregard I_1, I_2, I_3 in comparison with unity 1, we get

$$T_{\sigma\tau} = \lambda I_1 \cdot {}'g_{\sigma\tau} + 2\mu\varepsilon_{\sigma\tau}\;, \tag{3.14}$$

which gives us Hooke's law. If we take E as Young's modulus and ν as Poisson's ratio (Note: We can obtain any arbitrary law of elasticity depending on how we take ϕ and I_1, I_2, I_3), we get the following relations among λ, μ and E, ν :

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)} .$$

IV. DEFORMATION OF THIN SHELLS AND THE EQUATIONS OF DYNAMICS

Below, assume constant thickness and consider very thin shells; call a curved surface (shell) whose thickness is divided evenly into two parts a "central surface". Use α, β, γ as the indices for the vector and tensor components relative to a central surface; we adopt the convention that their values are taken as 1 and 2. Furthermore, before deformation, we assume beforehand that R_1 is so chosen that the surfaces where ξ^3 = constant coincide with the central surfaces. The perpendicular line from a) the position of any particle $\mathfrak{P}$ before deformation, in the shell to b) the central surface before deformation is assumed to intersect the perpendicular to the central surface also after deformation. We affix the caret symbol " ^ " to the various quantities relating to the central surface, and distinguish from the general point quantities inside the shell. If we designate by z the distance measured along the normal, before deformation, to the central surface, then the distortion tensor of any arbitrary point within the shell becomes as follow

$$\left.\begin{aligned} \varepsilon_{\alpha\beta} &= \hat{\varepsilon}_{\alpha\beta} + \varepsilon'_{\alpha\beta} - z(\kappa_{\alpha\beta} + \lambda'_{\alpha\beta}) + z^2 \mu_{\alpha\beta} \\ \varepsilon_{3\alpha} &= \varepsilon_{\alpha 3} = \varepsilon'_{3\alpha} - z\lambda'_{3\alpha} \end{aligned}\right\} \quad (\alpha, \beta = 1, 2) \qquad (4.1)$$

where

$$\hat{\varepsilon}_{\alpha\beta} = {}'\hat{g}_{\alpha\beta} - \hat{g}_{\alpha\beta}$$

expresses the distortion of the central surface. Also, if we designate by w'_λ the components relative to the coordinate system, after deformation of the relative displacement vector corresponding to a point fixed on the central surface, we express the relative distortion with respect to the central plane by

$$\varepsilon'_{\sigma\tau} = \frac{1}{2}\left(w'_{\sigma,\tau} + w'_{\tau,\sigma} + {}'g_{\lambda\kappa}\, w^{\lambda}_{;\sigma}\, w^{\kappa}_{;\tau}\right) \;;$$

and the quantity relating the change in the curvature of the central plane is

$$\kappa_{\alpha\beta} = \frac{1}{\sqrt{g_{33}}} \cdot \left(w_{3,\alpha\beta} + \Gamma^{\mu}_{\alpha\beta}\, w_{\mu,3} + w_{\mu,\alpha\beta}\, w'^{\mu}_{,3}\right).$$

Again, designating by $h_{\alpha\beta}$ and $'h_{\alpha\beta}$ the second fundamental quantity of the central surface before and after deformation, respectively, we have:

Furthermore, designating by $\frac{1}{R_{(\alpha\beta)}}$ and $\frac{1}{'R_{(\alpha\beta)}}$ the radius of curvature before and after, respectively, deformation relative to the direction determined by the ratios of $d\xi^{\alpha},\ d\xi^{\beta}$ of the central surface, and by H and 'H the average curvatures and by K and 'K the total curvatures, respectively, we get the expression

$$\mu_{\alpha\beta} = \frac{1}{2}\left\{ {}'g_{\alpha\beta}\left(\frac{'H}{'R_{(\alpha\beta)}} - {}'K\right) - g_{\alpha\beta}\left(\frac{H}{R_{(\alpha\beta)}} - K\right)\right\}$$

(the usual summation relative to α, β does not hold).

From the assumptions regarding the relative distortion $\varepsilon'_{\sigma,\tau}$ with respect to the central surface, namely that the shell is very thin and that even after deformation the normal to the central surface is maintained, we can set $_{;3} = \sqrt{\frac{7}{g_{33}}} \cdot \frac{\partial}{\partial z}$ and also when disregarding terms in $w'_{\sigma,\tau}$ of the second degree, we can express w'_σ as a "nonrelative" function in ξ^3, relative only to $f, g, \mathfrak{h}, i$ and $\xi^1 \xi^2$, in the following approximate form:

$$\begin{cases} w'_1 = -z \cdot (f_{,1} + g_{,1}) + z^3 \mathfrak{h} \\ w'_2 = -z \cdot (f_{,2} + g_{,2}) + z^3 \mathfrak{h} \\ w'_3 = \frac{1}{\sqrt{g_{33}}} \cdot (zf + \frac{1}{3} z^3 i) = f + z^2 i \end{cases}.$$

Consequently, if we use $\varepsilon'_{3\alpha} = 0 \quad (\alpha = 1, 2)$ at the boundary condition $z = \pm h$ (the thickness of the shell is 2 h) in the above formula and both surfaces, we get

$$\begin{cases} \varepsilon'_{\alpha\beta} = (-z(f+g) + z^3 f)_{,\alpha\beta} \\ \varepsilon'_{3\alpha} = -\frac{1}{2}(g(1 - \frac{z^2}{h^2}))_{,\alpha} \qquad (\alpha, \beta = 1, 2). \\ \varepsilon'_{33} = 2zi \end{cases}$$

Consequently, we have

$$\int_{-h}^{h} \varepsilon'_{\alpha\beta} dz = \int_{-h}^{h} \varepsilon'_{33} dz = 0, \qquad (\alpha, \beta = 1, 2).$$

$$\int_{-h}^{k} z \varepsilon'_{3\alpha} dz = 0 \quad [sic]$$

(4.2)

Similarly,

$$\int_{-h}^{h} \lambda'_{\alpha\beta} dz = \int_{-h}^{h} \lambda'_{33} dz \qquad (\alpha, \beta = 1, 2)$$

$$\int_{-k}^{h} z \lambda'_{3\alpha} dz = 0$$

(4.3)

Again, setting
$$\begin{cases} \dfrac{\partial \xi^{\alpha}}{\partial \hat{\xi}^{\alpha}} \cdot \dfrac{\partial \xi^{\beta}}{\partial \hat{\xi}^{\beta}} \equiv k \quad \left(\dfrac{1}{2h} \displaystyle\int_{-h}^{h} k\,dz \doteq 1 \right) \\ \displaystyle\int_{-h}^{h} kT^{\alpha\beta} dz \equiv P^{\alpha\beta}, \quad \int_{-h}^{h} kzT^{\alpha\beta} dz = M^{\alpha\beta} \\ \displaystyle\int_{-h}^{h} k\rho\,dz \equiv \bar{\rho}, \quad \int_{-h}^{h} k\rho X^{\mu} = \bar{\rho}\cdot\bar{X}^{\mu} \\ \displaystyle\int_{-h}^{h} k\rho b^{\mu} dz \equiv \bar{\rho}\cdot b^{\mu} \end{cases}$$

and multiplying (3.7) by k and zh, we integrate over the shell's thickness interval; if we use (4.2) and (4.3), the equations of motion of the thin shell becomes
$$\begin{cases} P^{\lambda\mu}{}_{;\mu} + \bar{\rho}\cdot\bar{X}^{\lambda} - \bar{\rho}\cdot\bar{b}^{\lambda} = 0 \\ M^{\alpha\mu}{}_{;\mu} = P^{\alpha 3} \qquad\qquad (\alpha = 1, 2). \\ M^{3\mu}{}_{;\mu} = 0 \end{cases} \tag{4.4}$$

Since the shell is very thin, we can as usual set $P^{33} = 0$. Furthermore, if we use (4.2), we get
$$P^{\alpha\beta} = \frac{2hE\nu}{1-\nu^2}\left(\dot{I}'_1 g^{\alpha\beta} + \frac{1-\nu}{\nu}\bar{\varepsilon}^{\alpha\beta}\right), \tag{4.5}$$

where we take
$$\bar{\varepsilon}^{\alpha\beta} = \frac{1}{2h}\int_{-h}^{h} k\varepsilon^{\alpha\beta} dz$$

$$\dot{I}_1 = \bar{\varepsilon}^{\alpha}{}_{\alpha} \qquad (\alpha = 1, 2).$$

Again, if we set

$$\begin{cases} \hat{I}_1 = \hat{\varepsilon}^{\alpha}_{\alpha} \\ K_1 = \kappa^{\alpha}_{\alpha} \quad (\alpha = 1,2) \\ M_1 = \mu^{\alpha}_{\alpha} \end{cases} \qquad \begin{cases} \bar{I}'_1 = \frac{1}{2h}\int_{-h}^{h} \varepsilon^{\alpha}_{\alpha}\, dz \\ \bar{L}' = \frac{1}{2h}\int_{-h}^{h} \lambda'^{\alpha}_{\alpha}\, dz \quad (\alpha,\beta = 1,2) \\ \bar{\lambda}'^{\alpha\beta} = \frac{1}{2h}\int_{-h}^{h} \lambda'^{\alpha\beta}\, dz \end{cases}$$

we get

$$P^{\alpha\beta} = \frac{2hE\nu}{1-\nu^2}\left\{(\hat{I}_1 - 2h\bar{L}' + \frac{8}{3}h^3 M_1)g^{\alpha\beta} + \frac{1-\nu}{\nu}(\hat{\varepsilon}^{\alpha\beta} - 2h\bar{\lambda}'^{\alpha\beta} + \frac{3}{8}h^2\mu^{\alpha\beta})\right\}. \tag{4.6}$$

Similarly, if we set

$$\begin{aligned} \bar{\bar{\varepsilon}}'^{\alpha\beta} &= \frac{1}{4h^2}\int_{-h}^{h} z\,\varepsilon'^{\alpha\beta}\, dz \\ \bar{\bar{I}}'_1 &= \frac{1}{4h^2}\int_{-h}^{h} z\,\varepsilon'^{\alpha}_{\alpha}\, dz \end{aligned} \qquad (\alpha,\beta = 1,2)$$

then we get

$$M^{\alpha\beta} = \frac{2h^3}{3}\cdot\frac{E\nu}{1-\nu^2}\left\{\left(\frac{6\bar{\bar{I}}'_1}{h} - K_1\right)g^{\alpha\beta} + \frac{1-\nu}{\nu}\left(\frac{6\bar{\bar{\varepsilon}}'^{\alpha\beta}}{h} - \kappa^{\alpha\beta}\right)\right\}. \tag{4.7}$$

In (4.6), the term on the right side is a second-degree term in the curvature and is appended for the first time. However, this quantity is so small that one can usually set it together with $\varepsilon'^{\alpha\beta}$ and $\lambda'^{\alpha\beta}$ equal to zero in comparison with the other quantities. The quantities (4.6) which must be taken anew into special consideration as factors influencing finite distortion are the quantities $w^{3}_{,\alpha} w_{3,\beta}$ contained in $\hat{\varepsilon}^{\alpha\beta}$.

V. ENERGY OF DEFORMATION OF THIN SHELLS, AND CONDITIONS GOVERNING STABILITY

Since, in the case where the shell is thin, we can approximately set $p^{33} = 0$, we can express ε^3_3 in terms of ε^1_1 and ε^2_2 ; furthermore, if we disregrad $\varepsilon'_{\sigma\tau}$ and $\lambda'_{\sigma\tau}$ in equation (4.1), the deformation energy in the entire body of the shell, $\Phi = \int_V \rho\phi dV$, finally becomes as follows if we disregard the variation in density before and after deformation:

$$\Phi = A\int_S (\hat{I}_1^2 + 2(1-\nu)\hat{I}_2)dS + D\int_S (K_1^2 + 2(1-\nu)K_2)dS$$
$$= \frac{h}{E}\int_S (P_1^2 + 2(1+\nu)P_2)dS + \frac{1}{D}\int_S (M_1^2 + 2(1+\nu)M_2)dS, \qquad (5.1)$$

where $A = \dfrac{Eh}{2(1-\nu^2)}$, $\quad D = \dfrac{2Eh^3}{3(1-\nu^2)}$

$$\hat{I}_1 = \varepsilon^\alpha_\alpha \qquad \hat{I}_2 = \det \varepsilon^\beta_\alpha$$
$$K_1 = \kappa^\alpha_\alpha \qquad K_2 = \det \kappa^\beta_\alpha \qquad (\alpha, \beta = 1,2).$$
$$P_1 = P^\alpha_\alpha \qquad P_2 = \det P^\beta_\alpha$$
$$M_1 = M^\alpha_\alpha \qquad M_2 = \det M^\beta_\alpha$$

If we designate by Ψ the work done by external pressure p, we have

$$\Psi = p\int_S \mathring{w}_3\, dS \qquad (5.3)$$

/Note: (5.2) was omitted in the original./

where $\overset{\circ}{w}_3 = \sqrt{'g_{33}} \cdot w^3$

$$dS = \sqrt{\det {'g}^{\beta}_{\alpha}} \cdot d\xi^1 d\xi^2 \qquad (\alpha, \beta = 1, 2) .$$

Consequently, the energy Π of displacement of the entire body of the thin shell can be written as follows:

$$\Pi = \Phi - \Psi . \tag{5.4}$$

Therefore, the conditions governing the equilibrium of a shell for any arbitrary δw^{λ} is

$$\delta \Pi = 0 . \tag{5.5}$$

This equilibrium condition depends upon either stable equilibrium or unstable equilibrium and for any δw_{λ} [sic] it is necessary that $\delta^2 \Pi > 0$ or $\delta^2 \Pi < 0$. The load which causes a change from stable equilibrium to unstable equilibrium, relative to (5.5) and any δw^{λ}, is sought from the second equation:

$$\delta^2 \Pi = 0 . \tag{5.6}$$

IV. THE BASIC EQUATIONS OF DURCHSCHLAG

We have discussed the significance of Durchschlag in the Introduction. In order to obtain the relation between actual load and bend, one may take (4.6), (4.7), and (4.4) as the solution formulas after proper adjustments and solve in accordance with suitable boundary conditions. In the phenomenon of Durchschlag, however, the calculations are simple if carried out as described below.

Durchschlag occurs in either rods or shells which possess small initial curvature. For example, in the partial spherical shell [2] shown in Figure 3, if we assume no flexural rigidity even position (3) can maintain the state of equilibrium without external pressure. Furthermore, if we have the situation (1) and no initial distortion, then situation (3) too should have no distortion. If there is no distortion, then any point P in (1) should correspond to point Q of (3) on the straight line parallel to the Z-axis. Next, if we consider the flexural rigidity and subject such a position as (1) to an external pressure, we get a deformation such as (4); furthermore, a distortion is probably created in the center surface. However, if the point P is assumed to move to some such point as Q' in the state (4), then we can consider that, since the shell offers great resistance toward tension (elongation), the horizontal distance QQ' is very small. Therefore, from calculations in which it is assumed that all particles on (1) move parallel to the Z-axis it is easy to arrive at calculations which take the horizontal distance QQ' into consideration. Karman[2]) has already carried out computations with the assumption that the horizontal distance QQ' equals O. Since the initial curvature is small, we can as a first approximation consider the stress of the center surface and direction of distortion in the direction of the X-axis. In such a case, if we take $\xi^3 = 0$ as the base plane and designate by $W(\xi^1, \xi^2)$ the distance from the shell's initial base plane and by $\mathfrak{W}$ the perpendicular component of the bend, then the components of the distortion in the horizontal direction relative to of the central surface's distortion are:

$$\varepsilon_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha} + \frac{1}{2}(W + \mathfrak{W})_{,\alpha}(W + \mathfrak{W})_{,\beta} - \frac{1}{2}W_{,\alpha}W_{,\beta} \qquad (6.1)$$

$$(\alpha, \beta = 1, 2)$$

In the above equation, W and $\mathring{\mathfrak{W}}$ are invariant with respect to transformations of curvilinear coordinates within the base plane. If we employ the conditions $W = \mathring{\mathfrak{W}} = 0$ on the boundary, equations (5.4) and (5.5) can be written as below:

$$\delta\Pi = \delta(\Phi - \Psi) = \int_S P^{\alpha\beta}{}_{,\beta}\,\delta(\mathring{w}_{,\alpha} + \mathring{\mathfrak{W}}_{,\alpha}) + \{ D g^{\alpha\beta} g^{\gamma\delta} \mathring{\mathfrak{W}}_{,\alpha\beta\gamma\delta} - P^{\alpha\beta}(W + \mathring{\mathfrak{W}})_{,\alpha\beta} - p \}\,\delta\mathring{\mathfrak{W}} = 0$$
$$(\alpha, \beta, \gamma, \delta = 1, 2).$$

Consequently we have:

$$P^{\alpha\beta}{}_{,\beta} = 0, \qquad D\Delta\Delta\mathring{\mathfrak{W}} = p + P^{\alpha\beta}(W + \mathring{\mathfrak{W}})_{,\alpha\beta} \qquad (\alpha, \beta = 1, 2). \tag{6.2}$$

The corresponding cond tions become, instead of (2.8) and (6.1), as follows:

$$\varepsilon_{\alpha\beta,\gamma\delta} + \varepsilon_{\gamma\delta,\alpha\beta} - \varepsilon_{\alpha\gamma,\beta\delta} - \varepsilon_{\beta\delta,\alpha\gamma} = (W + \mathring{\mathfrak{W}})_{,\alpha\gamma} \cdot (W + \mathring{\mathfrak{W}})_{,\beta\delta} - (W + \mathring{\mathfrak{W}})_{,\alpha\beta} \cdot (W + \mathring{\mathfrak{W}})_{,\gamma\delta}$$
$$- (W_{,\alpha\gamma} \cdot W_{,\beta\delta} - W_{,\alpha\beta} \cdot W_{,\gamma\delta}). \tag{6.3}$$

Next we try to rewrite the above equation for special cases

(1) RECTANGULAR COORDINATE SYSTEM

$$(\xi^1 = x,\ \xi^2 = y,\ [\xi^3 = z];\ \mathring{w}^1 = u,\ \mathring{w}^2 = v,\ \mathring{\mathfrak{W}} = w)$$

Distortion components: *

$$\begin{cases} \overset{\circ}{\varepsilon}_{11} = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial W}{\partial x} + \dfrac{\partial w}{\partial x}\right)^2 - \dfrac{1}{2}\left(\dfrac{\partial W}{\partial x}\right)^2 \\[2ex] \overset{\circ}{\varepsilon}_{22} = \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial W}{\partial y} + \dfrac{\partial w}{\partial y}\right)^2 - \dfrac{1}{2}\left(\dfrac{\partial W}{\partial y}\right)^2 \\[2ex] \overset{\circ}{\varepsilon}_{12} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) + \dfrac{1}{2}\left(\dfrac{\partial W}{\partial x} + \dfrac{\partial w}{\partial x}\right)\cdot\left(\dfrac{\partial W}{\partial y} + \dfrac{\partial w}{\partial y}\right) - \dfrac{1}{2}\dfrac{\partial W}{\partial x}\dfrac{\partial W}{\partial y}. \end{cases} \tag{6.3}$$

(*Note: We use the designation $\overset{\circ}{\varepsilon}_{\alpha\beta} = \varepsilon_{\alpha\beta}/g_{\alpha\beta}$ and designate the physical components relative to the basic coordinate system. We conform to this in what follows.)

Equations of equilibrium (we introduce the stress function F):

$$\overset{\circ}{P}_{11} = \frac{\partial^2 F}{\partial y^2}, \qquad \overset{\circ}{P}_{12} = -\frac{\partial^2 F}{\partial x \partial y}, \qquad \overset{\circ}{P}_{22} = \frac{\partial^2 F}{\partial x^2},$$

$$D\Delta\Delta w = p + \frac{\partial^2 F}{\partial y^2}\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) - \frac{\partial^2 F}{\partial x \partial y}\left(\frac{\partial^2 W}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y}\right) + \frac{\partial^2 F}{\partial x^2}\left(\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right); \qquad (6.4)$$

corresponding conditions:

$$\Delta\Delta F = \left(\frac{\partial^2 W}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) - \left\{\left(\frac{\partial^2 W}{\partial x \partial y}\right)^2 - \frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right\}$$

$$\text{where} \quad \Delta\Delta \equiv \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}. \qquad (6.4')$$

Law of elasticity:

$$\begin{cases} \overset{\circ}{\varepsilon}_{11} = \frac{1}{E}(\overset{\circ}{P}_{11} - \nu \overset{\circ}{P}_{22}) = \frac{1}{E}\left(\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right) \\ \overset{\circ}{\varepsilon}_{22} = \frac{1}{E}(\overset{\circ}{P}_{22} - \nu \overset{\circ}{P}_{11}) = \frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2}\right) \\ \overset{\circ}{\varepsilon}_{12} = \frac{1}{G}\overset{\circ}{P}_{12} = -\frac{2}{E}(1+\nu)\cdot\frac{\partial^2 F}{\partial x \partial y}. \end{cases}$$

(9.5)

In the various equations above, however, the influence of elongation (tension) of the coordinates in the direction of the base plane is small and therefore omitted. Furthermore, as mentioned before, the components of the various quantities in the direction of the central surface are replaced by the components in the direction of the base plane.

In order to take these influences also into consideration, for the components appearing in the various equations above, we may use $\overset{\circ}{P}_{\alpha\beta} = P'_{\alpha\beta}/{}'g_{\alpha\beta}$ as the physical components of $P_{\alpha\beta}$ for example, and for the Christoffel symbols we may employ ${}'\Gamma^{\gamma}_{\alpha\beta}$ instead of $\Gamma^{\gamma}_{\alpha\beta}$. With respect to the latter we may use

$${}'g_{\alpha\beta} = g_{\alpha\beta} + 2\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} + 2\varepsilon_{\alpha\beta}$$

rather than (2.5); and in the former we may employ instead of (2.7) the following expression:

ε [sic]

$${}'\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + g^{\gamma\delta}\left(\frac{\partial \varepsilon_{\delta\alpha}}{\partial \xi^{\beta}} + \frac{\partial \varepsilon_{\delta\beta}}{\partial \xi^{\alpha}} - \frac{\partial \xi_{\alpha\beta}}{\partial \xi^{\delta}}\right) - g^{\gamma\delta}\left(\frac{\partial g_{\delta\alpha}}{\partial \xi^{\beta}} + \frac{\partial g_{\delta\beta}}{\partial \xi^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial \xi^{\delta}}\right)$$

$$(\alpha, \beta, \gamma, \delta = 1, 2).$$

(2) POLAR COORDINATES $(\xi^1 = r,\ \xi^2 = \theta;\ \overset{\circ}{w}^1 = u,\ \overset{\circ}{w}^2 = v,\ W = w)$

Distortion components:

$$\overset{\circ}{\varepsilon}_{11} = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial W}{\partial r} + \frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2$$

$$\overset{\circ}{\varepsilon}_{22} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} + \frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial \theta} + \frac{1}{2}\frac{\partial w}{\partial \theta}\right)^2 - \frac{1}{2}\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right)^2$$

$$\overset{\circ}{\varepsilon}_{12} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{2}\left(\frac{\partial W}{\partial r} + \frac{\partial w}{\partial r}\right)\left(\frac{1}{r}\frac{\partial W}{\partial \theta} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) - \frac{1}{2}\frac{\partial W}{\partial r}\frac{1}{r}\frac{\partial W}{\partial \theta} \qquad (6.5)$$

Equations of equilibrium (we introduce the stress function F):

$$\overset{\circ}{P}_{11} = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}, \qquad \overset{\circ}{P}_{12} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right), \qquad \overset{\circ}{P}_{22} = \frac{\partial^2 F}{\partial r^2}$$

$$D\Delta\Delta w = p + \left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\left(\frac{\partial^2 W}{\partial r^2} + \frac{\partial^2 w}{\partial r^2}\right) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right)\left\{\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right\} + \frac{\partial^2 F}{\partial r^2}\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\right). \qquad (6.6)$$

Suitable Conditions:

$$\Delta\Delta F = \left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right\}^2 - \left(\frac{\partial^2 W}{\partial r^2} + \frac{\partial^2 w}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)$$
$$- \left[\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right)\right\}^2 - \frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right],$$

where (6.61)

Law of elasticity:

$$\left\{\begin{aligned} \overset{o}{\varepsilon}_{11} &= \frac{1}{E}\left(\overset{o}{P}_{11} - \nu \overset{o}{P}_{22}\right) = \frac{1}{E}\left\{\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - \nu\frac{\partial^2 F}{\partial r^2}\right\} \\ \overset{o}{\varepsilon}_{22} &= \frac{1}{E}\left(\overset{o}{P}_{22} - \nu \overset{o}{P}_{11}\right) = \frac{1}{E}\left\{\frac{\partial^2 F}{\partial r^2} - \nu\left(\frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\right\} \\ \overset{o}{\varepsilon}_{12} &= \frac{1}{G}\overset{o}{P}_{12} = -\frac{2}{E}(1+\nu)\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial \theta}\right). \end{aligned}\right. \qquad (6.7)$$

Also in the various equations above, we have disregarded the effect of elongation (tension) of the central surface. These corrections all can be found in a way completely identical to that discussed in (1).

VIII. CONCLUSION

The author, on the basis of the elasticity theory of finite deformations, has examined the fundamentals of the theory of shells. If we compare his results with previous theories in the form of a table, we get Table No. 1 below. Finally he has derived the fundamentals equations relating to Durchschlag; he has indicated them in the form of definite concrete formulas suitable for solutions. He wishes to defer the explanation of these formulas to another occasion.

Table No. 1

		General theory of infinitesimal displacement ↓	Love's theory of shells ↓	Karman's theory of flat shells ↓	General theory of finite deformations, and theory of shells ↓
Components of displacement in the distortion components →		up to 1st	up to 1st	up to 2nd	up to 2nd
Law of elasticity →		Hooke's law	Hooke's law	Hooke's law	arbitrary
Deformation of coordinate system	In direction of surface →	disregarded	disregarded	disregarded	disregarded or up to the 1st
	perpendicular to surface →	"	"	up to 1st	up to 1st or 2nd
Variation in the metric (effect of tension of central surface) →		"	"	disregarded	disregarded or up to 1st
Variation of the curvature →		--	up to 1st	up to 1st	up to 1st or 2nd
The limits applied →		limited to infinitesimal deformations	bending small in comparison with shell thickness	bendings several times the shell thickness	arbitrary
Peculiarity of the load bending	Cases other than buckling →	linear	linear	nonlinear	nonlinear
	In the case of buckling →	--	buckling load is determined as the eigenvalue of the equilibrium equation. Displacement is indeterminate	up to buckling loads, bend 0; thereafter, nonlinear	Same as Karman's

Finally I should like to take this opportunity to extend my deepest thanks to Professor Watanabe, who kindly gave me guidance while I was conducting this investigation.

(Appended Note: The part after Section V was emended after the lecture meeting in order to make future methods of solution convenient.)

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DISCUSSION

Chairman (Mr IGUCHI Tsuneo): Are there any questions or discussions on the report just heard? Is anyone ...? Since it seems that no one has any remarks, I should like to express a few words of thanks to the author. Mr SUHARA, as he just explained, has just started on the very fundamentals of the case where the deformations of thin shells are large, and has also established the necessary formulas of elasticity. I understand that the printing of this thesis involved extremely great difficulties. Of course the theory of shells is of importance to shipbuilders. For example, the present theory of beams does not apply well to the thin skin of ship hulls; it is being temporarily used. Analysis is being carried out in the various fields of this theory. It is wonderful that work is being done on fundamental researches in this connection. We can expect great future developments. We are grateful for this tremendously difficult and splendid report. I am sure all will wish to show their appreciation together by applauding. (all applaud).

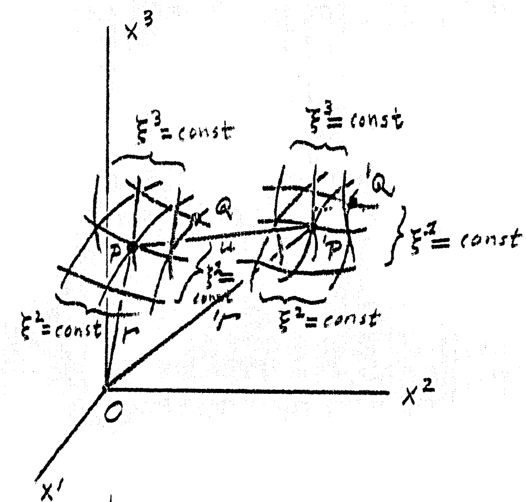

Figure 1

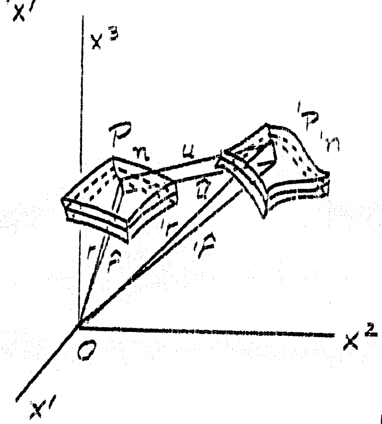

Figure 2

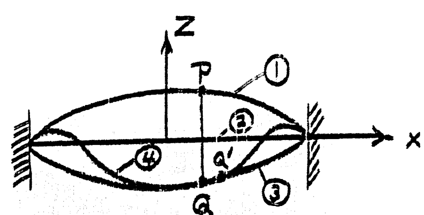

Figure 3

- E N D -

PROPER NAMES IN THE ORIGINAL

SUHARA Jiro	栖原 二郎
IGUCHI Tsuneo	井口 常雄
KAWANO Chūgi [Tadayoshi]	河野 忠義
MIZUKI Saburo	水木 三郎
SARATANI Yoshio	皿谷 嘉夫

Zōsen Kyōkai [Shipbuilding Association] 造船 協會

Zosen Kyokai Han-Shin Kurabu [Shipbuilding Association's Osaka-Kobe Club] 造船協會 阪神倶樂部

TECHNICAL TERMS IN THE ORIGINAL

zakutsu [Buckling]	挫屈
tenkutsu [Durchschlag]	轉屈

- END -