50X1-HUM

Title: CHARGE CONJUGATES FOR THE INVARIANT EQUATIONS OF GENERAL RELATIVITY (USSR) I. M. Gel'fand and A. M. Yaglom

Source: Zhurnal Eksperimental'noy i Teoreticheskoy Fiziki, Vol 18, No 12, 1948, pages 1105 - 1111

CONFIDENTIAL

CHARGE CONJUGATES FOR THE INVARIANT EQUATIONS OF GENERAL RELATIVITY

I. M. Gel'fand and A. M. Yaglom

[Note: The above is the title of an article written by I. M. Gel'fand and A. M. Yaylom, which appears in Zhurnal Eksperimental'noy i teoreticheskoy Fiziki Vol 18 No 12 (Dec 48), pages 1105-1111. The authors are affiliated with the Mathematical Institute of the Academy of Sciences USSR. The authors state that the results obtained in their earlier work, appearing in ZhETF 18, 703, 1948, are used in the present work, given below, to consider the problem concerning the concept of "charge conjugates", discussed earlier by E. Majorana in Nuova Cim 14, 171, 1937, and by N. A. Kramers in Proc Amst Acad Sci 40, 814, 1937, in connection with the invariant equations of general relativity. The results obtained in the present work are completely formulated and some pertinent examples are also given.]

1. Relativistically invariant equations describing the wave fields of elementary particles, we shall describe here, as in [1], in the following form:

$$L^k \frac{\partial \psi}{\partial x^k} + i\kappa\psi = 0 \tag{1}$$

where L^k ($k = 0, 1, 2, 3$) are finite or infinite dimensional matrices and kappa k is a real constant. Interaction with the external electromagnetic field be will introduced here in the usual way: when such a field is present equation - 1 is replaced by the following equation: $L^k \left(\frac{\partial}{\partial x^k} - i\varepsilon\varphi_k\right)\psi + i\kappa\psi = 0$ (2)

Here epsilon is the elementary charge of the particle corresponding (i.e. particle) to the field psi, ψ, and phi-sub-k, φ_k, is the four-dimensional potential of the electromagnetic field. (We shall make use here of the system of units in which Planck's h-bar equals one, $\hbar = 1$, and the velocity of light is unity, $c = 1$).

In the case where the equations describe particles with spin 0 and 1 (Deffine-Kemmer equations), the m trices L^k are usually so selected that they become real relative to reduction by i. In such a case, obviously, the starred psi function, ψ^*, complex conjugate to any arbitrary solution psi ψ of equation - 2, will satisfy the following equation:

$$L^k \left(\frac{\partial}{\partial x_k} + i \varepsilon \varphi_k \right) \psi^* + i \varkappa \psi^* = 0 \qquad (2')$$

which differs from equation-2 only in the substitution of plus epsilon, + ε, by minus epsilon, $-\varepsilon$. Equation-2' describes a particle with properties similar to those of the particle described by equation-2, but with the reverse sign of the charge (antiparticle). Equation-1 for its part, together with psi ψ, will always satisfy also the complex conjugate psi-star, ψ^*; that is, along with the solution corresponding to the particle under study, we shall always have a solution for the antiparticle also. For the neutral particle, it is essential to take into consideration that the particle and antiparticle coincide with each other; in this case we must describe neutral particles with spins 0 and 1 by means of the real functions $\psi = \psi^*$ satisfying equation-1.

Application of these considerations to the case of Dirac's theory was made by E. Majorana [2] and Kramers [3]. It is obviously impossible to consider in exactly the same way as in the case of spin 0 or 1; namely the transition from psi, ψ , to psi-star, ψ^*, in this case does not convert the solution of equation-2 to the solution of equation-2' and in general does not possess any relativistically invariant significance. Instead of the concept of complex conjugates Majorana and Kramers introduced for this case the new concept of charge conjugates. This concept and also the related theory of neutral particles for the case of spin 0, 1, or $1/2$ were also considered in [1-8]. The results obtained in [1], concerning the invariant equations of general relativity of the form of equation-1, permit one to consider without difficulty this problem in the general form. It will then be particularly clear that charge conjugates do generally esist in such cases. That charge conjugates do not exist in all cases is indicated in particular by examples of equations that describe particles with just one sign of the charge [9]; there also exist finite-dimensional equations for which there are no charge conjugates, as shown in section 7 below.

2. The wave function psi-upper-Q, ψ^Q, namely the charge conjugates to psi ψ, must satisfy the following conditions: (a) in Lorentz transformations psi-Q, ψ^Q, must transform just as psi ψ, and (b) for psi, ψ, satisfying equation-2 psi-Q, ψ^Q, must satisfy equation-2'.

The role of real wave functions in the theory of particles with spin 0 and 1 in this case will be played by the "neutral" wave functions for which psi equals psi-Q: $\psi = \psi^Q$; such functions should essentially be employed to describe neutral particles.

The components of the psi-Q function, ψ^Q, in all cases must be found in the form of linear transformed components of the psi-star function, ψ^*, namely the complex conjugates to psi, ψ. In other words if we use the designation

$$\psi^Q = Q\psi \,, \tag{3}$$

then the transformation Q will possess the following properties:

$$Q(\psi_1 + \psi_2) = Q\psi_1 + Q\psi_2, \text{ and } Q(\lambda\psi) = \lambda^* Q\psi \qquad (4), (5)$$

where lambda-star, λ^*, is the number which is complex conjugate to lambda, λ. The transformation possessing the properties (4) and (5), we shall call "antilinear".

For the Lorentz transformation we have $\psi \rightarrow \psi' = S\psi$ (6)

(the set of linear transformations S corresponding to all transformations of the Lorentz group forms the representation of this group). Condition (a) will be fulfilled if for this transformation we have

$$\psi^Q \rightarrow (\psi^Q)' = (\psi')^Q \,;$$

that is, if for all S the following equality holds:

$$SQ = QS \tag{7}$$

Now let psi, ψ , satisfy equation-2. Replacing in this equation psi, ψ , by $Q^{-1} \psi^{Q}$ and using equation-5 we easily see that psi-Q, ψ^{Q}, will satisfy the following equation:

$$-QL^{k}Q^{-1}(\partial/\partial x^{k} + i\varepsilon\varphi_{k})\psi^{Q} + i\kappa\psi^{Q} = 0 . \tag{8}$$

Thus in order to fulfil condition (b) it is necessary only to require that the following equality hold:

$$QL^{k}Q^{-1} = -L^{k}; \text{ that is, } QL^{k} + L^{k}Q = 0 \quad (k = 0, 1, 2, 3). \tag{9}$$

3. We note first of all that equation-7 will hold for all transformations S of the representation of a proper Lorentz group, only if this equality is fulfilled for infinitesimal transformations Ij^{k} of this representation; that is, if the following equalities hold:

$$QIj^{k} = Ij^{k}Q \quad (j \neq k;\ j, k = 0, 1, 2, 3). \tag{10}$$

Here only four equations of the six equations-10 will be independent: equations for I^{02} and I^{03} will be derived from the remaining equations by virtue of the interposable relations for Ij^{k}. By passing to equations-10 from transformations Ij^{k} to transformations H^{0}, H^{+}, H^{-}, F^{0}, F^{+}, F^{-} (see formula (2, 1) in [1]) and by using equation-5, we can rewrite these four independent equations in the following forms:

$$QH^{\circ} = H^{\circ}Q \tag{11}$$

$$QH^{+} = H^{-}Q \text{ and } QH^{-} = H^{+}Q \tag{12}$$

$$QF^{\circ} = F^{\circ}Q \tag{13}$$

In order that equation-7 should hold for all transformations S of the representation of the full Lorentz group, it is further necessary that the following equality should be fulfilled:

$$QT = TQ \tag{14}$$

(for the determination of T, see section 4 in [1].

Declassified in Part - Sanitized Copy Approved for Release 2012/06/08 : CIA-RDP82-00039R000100110053-1

We note further that by virtue of the relation $L^k = [L^\circ, L^{\circ k}]$ (k = 1, 2, 3) (see formula 3.1 in [1] and equations-10, the equations-9 for k = 1, 2, 3 follow from the following:

$$QL^\circ + L^\circ Q = 0 \tag{15}$$

Thus we need only seek an antilinear transformation Q which satisfies the relations-11, 12, 13, 14, and 15. This problem is very close in type to the problems considered in Sections 4 and 5 in [1]; its solution also can be effected in a manner that is entirely like that in the cases indicated.

4. In the space of psi functions, 4, we select as the coordinate vectors the same vectors xi-upper-k, lower-p-tau, $\xi^k_{p\tau}$, as were used in the work [1]. Let us take

$$Q\xi^k_{p\tau} = \sum_{k',p',\tau'} q^{k,k'}_{p\tau,p'\tau'}\, \xi^{k'}_{p'\tau'} \tag{16}$$

then, by virtue of equation-5 and the first of the formulas-2.2' in the work [1], it will follow from relation-11 that we have:

$$-ipq^{k,k'}_{p\tau,p'\tau'} = ip'q^{k,k'}_{p\tau,p'\tau'} \; ;$$

that is:

$$q^{k,k'}_{p\tau,p'\tau'} = \delta_{-p,p'}\, q^{k,k'}_{p\tau\tau'} \tag{17}$$

Further, by virtue of the succeeding two formulas -2.2' from [1], we can rewrite equation-12 relative to

$q^{k,k'}_{p-1,\tau\tau'}$ and $q^{k,k'}_{p\tau\tau'}$

in the form of the following two equations :

$$\sqrt{(k+p)(k-p+1)}\cdot q^{k,k'}_{p-1,\tau\tau'} + \sqrt{(k'+p)(k'-p+1)}\cdot q^{k,k'}_{p\tau\tau'} = 0$$

$$\sqrt{(k'+p)(k'-p+1)}\cdot q^{k,k'}_{p-1,\tau\tau'} + \sqrt{(k+p)(k-p+1)}\cdot q^{k,k'}_{p\tau\tau'} = 0 .$$

Hence (see the similar conclusions in [1], on pages 711 and 720) it follows that we have:

$$q^{k,k'}_{p\tau\tau'} = 0 \text{ for } k' \neq k, \text{ and } q^{k,k}_{p\tau\tau'} = -q^{k,k}_{p-1,\tau\tau'} \text{; that is,} \tag{18}$$
$$q^{k,k'}_{p\tau\tau'} = \delta_{k,k'} (-1)^{[p]} q^{k}_{\tau\tau'} .$$

Let us now use equation-13. By virtue of the first of formulas-2.2" from [1], this equation leads to the following three relations:

$$A^{\#}_k(\tau)\, q^k_{\tau\tau'} + A_k(\tau')\, q^k_{\tau\tau'} = 0 \tag{19}$$

$$B^{\#}_k(\tau)\, q^k_{\tau\tau'} - B_k(\tau') \text{ [illegible]} \tag{20'}$$

$$B_k(\tau)\, q^{k-1}_{\tau\tau'} - B^{\#}_k(\tau') \text{ [illegible]} \tag{20''}$$

These relations indicate (see the perfectly identical conclusion on page 720 of the work [1] that

$$q^k_{\tau\tau'} = \varepsilon^{(k)} q_{\tau\tau^*} \quad \text{for } \tau' = \tau^*$$
$$q^k_{\tau\tau'} = 0 \quad \text{for } \tau' \neq \tau^* \tag{21}$$

Here tau-star τ^* is the representation determined by the pair $(k_0, -k_1^*)$ — that is, for tau τ determined by the pair (k_0, k_1); epsilon-upper-k, $\varepsilon^{(k)}$ equals plus or minus one ($\varepsilon^{(k)} = \pm 1$), and is defined by the following recurrence formula:

$$\varepsilon^{(k)} = \varepsilon^{(k-1)} B^{\#}_k(\tau)/B_k(\tau^*) \tag{22}$$

(see formula-5.21 in [1]; and finally $q_{\tau\tau^*}$ is an arbitrary complex number. (Note: In connection with the number pair $(k_0, -k_1^*)$ here, one can, generally speaking introduce in the equation several representations not cited, which are defined by one and the same pair $(k_0, -k_1^*)$. However, in order not to complicate the formulas, we will here assume therefore that

CONFIDENTIAL

each nonderived representation is introduced in the equation not more than once, so that tau-star τ^* is unique. Combining formulas-17, 18, and 21, we will finally have:

$$q^{k,k'}_{p\tau,p'\tau'} = \delta_{-p,p'} \, \delta_{\tau^*,\tau'} \, (-1)^{[p]} \, \varepsilon^{(k)} \, q_{\tau\tau^*} . \qquad (23)$$

(Note: Equation-23 assumes a particularly simple form in the finite-dimensional case: here epsilon-k $\varepsilon^{(k)}$ can always be considered equal to $(-1)^{[k]}$, and equation-23 is converted into the following:

$$q^{k,k'}_{p\tau,p'\tau'} = \delta_{-p,p'} \, \delta_{k,k'} \, \delta_{\tau^*,\tau'} \, (-1)^{p+k} \, q_{\tau\tau^*} . \qquad (23)$$

Here tau-star τ^* is defined by the pair $(k_0, -k_1)$ and consequently coincides with tau τ; see Section 4 in [1]).

In as much as we consider here that tau-star τ^* is determined uniquely relative to tau τ, then we can omit the index tau-star τ^* in the expression $q_{\tau\tau^*}$ and hence write it as follows:

$$q_{\tau\tau^*} = q(\tau) .$$

Thus with each representation tau, τ, entering in the equation we juxtapose the number $q(\tau)$ for comparison; in order that the numbers $q(\tau)$ should yield charge conjugates, it is necessary that for all pairs τ, τ^* of "coupled" representations the following equation should hold:

$$q(\tau') = - q(\tau) \, c_{\tau^* \tau^{*\prime}} / c^*_{\tau\tau'} . \qquad (27)$$

By virtue of formula-5.32 in the work [1] for equations obtained from the Lagrange function, we represent equation-27 in the following form:

$$q(\tau') = - q(\tau) \, c^*_{\tau'\tau} \, a(\tau) / c^*_{\tau\tau'} \, a(\tau') . \qquad (28)$$

(We note that $a(\tau) = a_{\tau\tau^*}$ are the coefficients of an invariant bilinear form).

CONFIDENTIAL

6. Thus the operator Q of charge conjugation in all cases is determined by formulas-16 and 23, where the numbers $q_{\tau\tau^*} = q(\tau)$ must satisfy condition-27 (or, for equations obtained from the Lagrange function, q must satisfy condition-28). By assigning arbitrarily the number $q(\tau)$ for a certain tau, τ, we can determine $q(\tau')$ according to formula-27 (or 28) for all tau-primes, τ', that are coupled with tau, τ. In the case of a non-resolving or non-decomposing equation we can consequently determine $q(\tau)$ for all tau, τ. In order that this determination should be unique and not lead to contradictions (that is, in order that charge conjugates should exist), it is necessary that we should obtain for it the original value of $q(\tau)$ each time that we should, relative to the chain of coupled representations, return back to the same representation tau, τ, with which we started. From formula-28 it follows that in the case of equations obtained from the Lagrange equation this condition will be fulfilled if for any closed chain of pair-wise coupled representations, namely:

$$\tau \rightleftarrows \tau' \rightleftarrows \tau'' \rightleftarrows \cdots\cdots \tau^{(n)} \rightleftarrows \tau ,$$

the number $C_{\tau\tau'}$ determining the equation will satisfy the following condition:

$$C_{\tau\tau'}\ C_{\tau'\tau''}\ \ldots\ldots C_{\tau^{(n)}\tau} / C_{\tau'\tau}\ C_{\tau''\tau'}\ \ldots\ C_{\tau\tau^{(n)}} = (-1)^n . \quad (29)$$

If even for just one such chain (29) it does not hold true, then this condition will not be fulfilled. Thus charge conjugates will exist only when condition-29 is fulfilled.

In order that charge conjugates should be invariant relative to reflections (that is, in order that psi-Q, ψ^Q, during reflections should transform in the same way as psi, ψ), it is further necessary that condition-24 (or 24')should be fulfilled; that is, it is necessary that we have:

$$C_{\tau\tau'}\ C_{\tau'\tau''}\ \ldots\ C_{\tau^{(m)}\dot{\tau}} / C_{\tau'\tau}\ C_{\tau''\tau'}\ , \ldots\ C_{\dot{\tau}\tau^{(m)}} = (-1)^m \quad (30)$$

or

$$C_{\tau\tau'} C_{\tau'\tau''} \dots C_{\tau^{(m)}\dot{\tau}} / C_{\tau'\tau} C_{\tau''\tau'} \dots C_{\dot{\tau}\tau^{(m)}} = (-1)^{m+1} \qquad (30')$$

(in the case of half-integral spin and transformation T satisfying equation-25).

If the representations tau and tau-dot, τ and $\dot{\tau}$, entering in the equation cannot be joined by a chain of coupled representations (that is, equation-1, just as the equation invariant relative to the proper group, is resolved or decomposed, and its decomposition is due only to the presence of transformation T which connects tau, τ, with tau-dot, $\dot{\tau}$. See page 718 in [1], then assigning q(τ) arbitrarily we have q($\dot{\tau}$) necessarily determined from condition-24 (or, if convenient, 24'). Here the charge conjugates (if they do exist) will certainly be always invariant relative to reflections. Exactly similarly, if tau, τ, and tau-star, τ^*, cannot be joined by a chain of coupled representations (non-decomposibility of the equation is assured only by the presence of an invariant form; see page 723 in [1], then by assigning q(τ) arbitrarily we must have q(τ^*) so determined that the following condition would be fulfilled

$$\psi^{QQ} = \psi \quad \text{(that is, } Q^2 = E\text{)}. \qquad (31)$$

(in the case where τ and τ^* can be joined in a chain of coupled representations, then condition-31 can turn out to be fulfilled or not fulfilled; we cannot require that this condition should hold).

7. We note that in equation-29 n can be odd only in the case E, according to the classification given in the work [9]. In particular, to the examples of equations shown in [9], which (i.e. examples) describe particles with charge of one sign, corresponds one unique representation of tau, τ, which is coupled with itself, obviously in this case charge conjugates do not exist (according to equation-27 here it necessary that we should have q(τ) = -q(τ).

For a binomial closed chain of coupled representations, namely

$$\tau \rightleftarrows \tau' \rightleftarrows \tau$$

condition-29 will always be fulfilled; consequently charge conjugates will exist for equations not possessing closed chains of representations of a more complex form (particularly for equations of Dirac and Deffine-Kenmer). It is also easy to verify that in the case of Deffine-Kemmer equations charge conjugates will be invariant relative to reflection; in the case of Dirac equations, charge conjugates will be invariant relative to reflections only when condition-25 is fulfilled (see [2, 4]).

As an example of the application of the above-obtained formulas to equations for which earlier charge conjugates were not considered, let us construct here the charge conjugate for a particle with spin $^3/2$ (for the corresponding equation see, for example, Section 7 in [1]). We shall also solve the more general problem; relative to equation-1 we shall only require that the "chain scheme" for it should have the form:

$$\begin{array}{ccc} 1 & \rightleftarrows & 2 \\ \downarrow\uparrow & & \downarrow\uparrow \\ \dot{1} & & \dot{2} \end{array}$$

where $1^* = \dot{1}$ and $2^* = \dot{2}$. The matrix of numbers $C_{\tau\tau'}$ here can be reduced, depending upon the form of the invariant formula, to one of the following:

(a) $\begin{pmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & \beta \\ \beta & 0 & 0 & \gamma \\ 0 & \beta & \gamma & 0 \end{pmatrix}$ or (b) $\begin{pmatrix} 0 & \alpha & \beta & 0 \\ \alpha & 0 & 0 & \beta \\ -\beta & 0 & 0 & \gamma \\ 0 & -\beta & \gamma & 0 \end{pmatrix}$

where alpha and gamma (α , γ) are real and beta, β , is real positive (see pages 729, 730 in [1]. Condition-29 obviously will be fulfilled in both cases so that charge conjugates will exist here. Setting $q(1) = q$, we will obtain according to formula-27 for the remaining tau, τ , the following relations:

(a) $q(1) = q$, $q(\dot{1}) = -q$, $q(3) = -q$, $q(\dot{3}) = q$;

(b) $q(1) = q$, $q(\dot{1}) = -q$, $q(3) = q$, $q(\dot{3}) = -q$

Transformation Q is completely determined in the same manner. As in the case of Dirac's equation, the transformation turns out to be invariant relative to reflections only when condition-25 is **fulfilled**; if we assume moreover that $q(1) = 1$, then condition-31 also will be fulfilled.

In conclusion let us deduce the "chain scheme" that determines the finite-dimensional relativistically invariant equation for which charge conjugates cannot exist. The following, for example, will be such a scheme:

$$\begin{array}{ccccccc}
(-\frac{3}{2}, \frac{5}{2}) & \rightleftarrows & (-\frac{1}{2}, \frac{5}{2}) & \rightleftarrows & (\frac{1}{2}, \frac{5}{2}) & \rightleftarrows & (\frac{3}{2}, \frac{5}{2}), \\
\downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \\
(-\frac{3}{2}, \frac{7}{2}) & \rightleftarrows & (-\frac{1}{2}, \frac{7}{2}) & \rightleftarrows & (1/2, \frac{7}{2}) & \rightleftarrows & (\frac{3}{2}, \frac{7}{2}).
\end{array}$$

In order that charge invariance should exist here, we must require that the numbers $C_{\tau\tau'}$, should satisfy the following **relations**:

$$C_{12}C_{23}C_{34}C_{41} = C_{14}C_{43}C_{32}C_{21},$$

where

$$1 \sim (-\frac{3}{2}, \frac{5}{2}),\ 2 \sim (-\frac{1}{2}, \frac{5}{2}),\ 3 \sim (-\frac{1}{2}, \frac{7}{2}),\ 4 \sim (-\frac{3}{2}, \frac{7}{2}).$$

In the opposite case charge conjugates will not exist.

Submitted 25 June 1948.

Bibliography

1. I. M. Gel'fand and A. M. Yaglom. ZhETF, 18, 703, 1948.
2. E. Majorana. Nuovo Cim, 14, 171, 1937.
3. H. A. Kramers. Proc Arrnst Acad Sci, 40, 814, 1937.
4. G. Fochah. Nuovo Cim, 14, 322, 1937.
5. W. H. Furry. Phy Rev, 54, 56, 1938.
6. F. J. Bellingante. Physica, 6, 849, 870, 1939.
7. W. Pauli and F. J. Bellinfante. Physica, 7, 177, 1940.
8. W. Pauli. Relativistic theory of Elementary Particles (translated into Russian, 1947 at Moscow).

9. I. M. Gel'fand and A. M. Yaglom. ZhETF, Vol 18, No. 12 (Dec 1948), pp. 1096-1104. (Note: [9] is an article appearing in the same issue as the above article; in fact [9] immediately precedes the one just given here.)

- E N D -