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CONCERNING THE INVARIANT CONSTRUCTION OF THE QUANTUM THEORY OF A FIELD

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The repeated attempts to construct a quantum theory of a field whose relativistic invariancy would not be purely illusory and in which it would not be necessary to resort to physically absurd consequences with the help of the so-called "deductive formalism" have not met with success. Therefore, we have undertaken in this work a detailed investigation of the conditions imposed on any quantum theory of a field solely by the requirement of relativistic invariancy. We write these conditions in the form of the well-known permutation relationships:

$$a)\ [H P_\alpha] = 0 \qquad b)\ [P_\alpha P_\beta] = 0 \qquad c)\ [I_\alpha H] = iP_\alpha$$

$$d)\ [I_\alpha P_\beta] = i\delta_{\alpha\beta} H \qquad e)\ [I_\alpha I_\beta] = -i e_{\alpha\beta\gamma} M_\gamma \qquad f)\ [I_\alpha M_\beta] = i e_{\alpha\beta\gamma} I_\gamma \qquad (1)$$

$$g)\ [M_\alpha M_\beta] = i e_{\alpha\beta\gamma} M_\gamma \qquad h)\ [P_\alpha M_\beta] = i e_{\alpha\beta\gamma} P_\gamma \qquad i)\ [M_\alpha H] = 0 \quad (\alpha, \beta, \gamma = 1, 2, 3)$$

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Here H, P_α, M_β and I_γ are the operators, respectively, of energy, momentum, moment, and the time part of the 4-moment ($\hbar = c = 1$).

It is easily seen that all the relationships (1) are not independent. For example, all remaining relationships are consequences of the relationships (1a) to (1f). Therefore, we will consider only the relationships of the first group.

The relationships (1) were recently studied by Dirac (P. A. M. Dirac, Rev. Mod. Phys., 21, 392, 1949) for the case of the classical theory. We will consider them within the limits of the method of secondary quantization. We introduce the ordinary operators a^*_μ, a_λ, conforming to the permutation rules:

$$a_\lambda a^*_\mu - a^*_\mu a_\lambda = \delta(\mu - \lambda) \qquad (2)$$

(for simplicity, we restrict ourselves to the case of Bose statistics). The index μ is a combination of signs m and M, the first of which is the ordinary wave vector, while the second characterizes the nature of the field, the charge, the number of the component, etc. In those cases in which M takes on discrete values, we employ the symbol $\int dM(\cdots)$. We designate

$$\int (dm)_3 \, dM(\cdots) = \int d\mu(\cdots) \quad \int d\mu_1 \ldots d\mu_m(\cdots) = \int dm(\cdots) \tag{3}$$

$$\prod_{i=1}^{m} a^*_{\mu_i} = \Pi^*_m \quad \prod_{k=1}^{\ell} a_{\lambda_k} = \Pi_\ell \tag{4}$$

We will also use the symbols ( ) and [ ] to designate symmetry and alternation, respectively, with respect to the indices in the parentheses or brackets.

We will consider that any operator encountered in the theory can be expressed through a^*_μ, and a_λ and consequently can be written in the form:

$$F = \sum_{n=1}^{\infty} \sum_{t=1}^{\infty} \int dn \, dt \, K_{nt}(\nu_1 \ldots \nu_n ; \tau_1 \ldots \tau_t) \int \Pi^*_n \Pi_t \tag{5}$$

where the functions $f_{ml}(\mu_1, \ldots, \mu_m ; \lambda_1, \ldots \lambda_\ell) = f_{ml}(\mu ; \lambda)$ are symmetric with respect to permutations of the arguments within each group. We can show that the commutator of two operators F and G of the type (5) has the form:

$$FG - GF = \sum_{n=1}^{\infty} \sum_{t=1}^{\infty} \int dn \, dt \, K_{nt}(\nu_1 \ldots \nu_n ; \tau_1 \ldots \tau_t) \int \Pi^*_n \Pi_t \tag{6}$$

where

$$K_{nt}(\nu ; \tau) = \sum_{s=1}^{\infty} \int ds \sum_{m=0}^{n} \sum_{\ell=0}^{t} \frac{(n+s-m)!(t+s-\ell)!}{(n-m)!(t-\ell)!} \times$$

$$\times \{ f_{m,t+s-\ell}(\nu_1 \ldots \nu_m ; \tau_1 \ldots \tau_{t-\ell} ; \sigma_1 \ldots \sigma_s) \times$$

$$\times g_{n+s-m,\ell}(\nu_{m+1} \ldots \nu_n ; \sigma_1 \ldots \sigma_s ; \tau_{t-\ell+1} \ldots \tau_t) \times$$

$$\times - g_{m,t+s-\ell}(\nu_1 \ldots \nu_m ; \tau_1 \ldots \tau_{t-\ell} ; \sigma_1 \ldots \sigma_s) \times \tag{7}$$

$$\times f_{n+s-m,\ell}(\nu_{m+1} \ldots \nu_n ; \sigma_1 \ldots \sigma_s ; \tau_{t-\ell+1} \ldots \tau_t) \}$$

In conformity with the above, we write the operators P_α, H, I_β in the form:

$$P_\alpha = \int d\mu \, d\lambda \, \delta(\mu - \lambda) p_\alpha(\mu) a^*_\mu a_\lambda \tag{8a}$$

$$H = \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \int dm \, d\ell \, f_{m\ell}(\mu, \lambda) \Pi^*_m \Pi_\ell \tag{8b}$$

$$I_\beta = \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \int dm \, d\ell \, I^\beta_{m\ell}(\mu, \lambda) \Pi^*_m \Pi_\ell \tag{8c}$$

where $p_\alpha(\mu) = E(M) m_\alpha$, and $E(M) = \pm 1$ according to the sign of the charge.

From the relationships (1a) and (1d), we easily obtain:

$$f_{m,\ell}(\mu, \lambda) = \delta\left(\sum_{i=1}^{m} p(\mu_i) - \sum_{k=1}^{\ell} p(\lambda_k)\right) \varphi_{m,\ell}(\mu ; \lambda) \tag{9}$$

$$I^\beta_{m\ell}(\mu ; \lambda) = -i \delta'_\beta \left(\sum_{i=1}^{m} p(\mu_i) - \sum_{k=1}^{\ell} p(\lambda_k)\right) \varphi_{m,\ell}(\mu ; \lambda) \tag{10}$$

(where the prime indicates differentiation with respect to the entire argument).

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From (1c) it follows that, because of (7), (8), (9), and (10):

$$\delta\Big(\sum_{i=1}^{n} p(\nu_i) - \sum_{k=1}^{\ell} p(\tau_k)\Big)\Psi^{\alpha}_{(n)(\ell)}(\nu;\tau) = \delta_n\,\delta_\ell\,E(M)\,\delta(\nu-\tau)\,n_\alpha$$

$$\Psi^{\alpha}_{n\ell}(\nu;\tau) = \sum_{s=1}^{\infty}\int ds\sum_{m=0}^{n}\sum_{\ell=0}^{\ell}\frac{(n+s-m)!\,(\ell+s-\ell)!}{(n-m)!\,(\ell-\ell)!\,s!}\,\delta(s^{*}-S_s)\times \qquad (11)$$

where

$$\times\frac{1}{E(S_s)}\frac{1}{\partial(S_s)_\alpha}\Big[\varphi_{m,\ell+s-\ell}(\nu;\tau,\sigma)\,\varphi_{n+s-m,\ell}\Big]$$

$$S^{*} = \frac{1}{E(\zeta_s)}\Big\{\sum_{i=1}^{m} p(\nu_i) - \sum_{k=1}^{\ell-\ell} p(\tau_k) - \sum_{j=1}^{s-1} p(\sigma_j)\Big\} \qquad (12)$$

On the basis of (1e), (8c), and (10), we have, taking (7) and (11) into consideration

$$M_\gamma = i\,e_{\alpha\beta\gamma}\Big\{\int d\mu\, d\lambda\,\delta'_{\beta}\,(p(\mu)-p(\lambda))\,m_\alpha\,E(M)\,a^{*}_{\mu}a_\lambda\; +$$

$$+\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty}\int d\mu\int d\lambda\,\delta\Big(\sum_{i=1}^{m} p(\mu_i) - \sum_{k=1}^{\ell} p(\lambda_k)\Big)h^{\alpha\beta}_{m\ell}(\mu,\ell)\,\overline{\Psi^{*}_{m}}\,\Psi_{\ell}\Big\} \qquad (13)$$

where

$$h^{\alpha\beta}_{m,\ell}(\mu,\lambda) = \frac{1}{\ell}\sum_{k=1}^{\ell}\frac{\partial}{\partial(\ell_k)_\beta}\Big[\Psi^{\alpha}_{m\ell}(\mu;\ell)\,\frac{1}{E(\lambda_k)}\Big] \qquad (14)$$

Finally, the relationship (1f) yields:

$$\lambda\sum_{s=1}^{\infty}\int ds\sum_{m=0}^{n}\sum_{\ell=0}^{\ell}\frac{(n+s-m)!\,(\ell+s-\ell)!}{(n-m)!\,(\ell-\ell)!\,s!}\Big\{\Big[\delta(S^{*}-S_s)\frac{\partial}{\partial(S_s)_\gamma}\big(\varphi_{m,\ell+s-\ell}\,\Psi^{\ell\alpha}_{n-s-m,\ell}\big)+$$

$$+\,\delta(S_s-S^{*})\frac{1}{\partial(S_s)_\gamma}\big(\varphi_{n+s-m,\ell}\,\Psi^{\ell\alpha}_{m,\ell+s-\ell}\big)\Big]\delta'_{\beta}\Big]\Big(\sum_{i=1}^{n} p(\nu_i) - \sum_{k=1}^{\ell} p(\tau_k)\Big) -$$

$$-\frac{1}{E(\zeta_s)}\big[\delta(S_s-S^{**})\,\varphi_{n+s-m}\,\Psi^{\ell\alpha}_{m,\ell+s-1} - \delta(S^{*}-S_s)\,\varphi_{m,\ell+s-\ell}\,\Psi_{n+s-m,\ell}\big]\times$$

$$\times\,\delta''_{\beta]\gamma}\Big(\sum_{i=1}^{n} p(\nu_i) - \sum_{k=1}^{\ell} p(\tau_k)\Big) = \delta_{\beta\gamma}\,\delta'_{\alpha}\Big(\sum_{i=1}^{n} p(\nu_i) - \sum_{k=1}^{\ell} p(\tau_k)\Big)\varphi_{n\ell}(\nu,\tau) -$$

$$-\,\delta_{\alpha\gamma}\,\delta'_{\beta}\Big(\sum_{i=1}^{n} p(\nu_i) - \sum_{k=1}^{\ell} p(\tau_k)\Big)\varphi_{n\ell}(\nu;\tau) \qquad (15)$$

where

$$S^{*} = \frac{1}{E(S_s)}\Big\{\sum_{k=\ell-\ell+1}^{\ell} p(\tau_k) - \sum_{i=m+1}^{n} p(\nu_i) - \sum_{j=1}^{s-1} p(\sigma_j)\Big\} \qquad (16)$$

The solution of equations (11) and (15) gives all forms of Hamiltonians which are possible in the relativistically invariant theory. Equation (11) plays a basic role, since from this equation are determined the values of $\phi_{ml}(\mu;\lambda)$ on the special surface $\sum_{i=1}^{m} p(\mu_i) = \sum_{k=1}^{\ell} p(\lambda_k)$ which figure in the Hamiltonian. Equation (11) is easily solved in the general form only for free fields, when all functions $\phi_{ml}(\mu;\lambda) = 0$, except $\phi_{11}(\mu;\lambda)$. In this case, the Hamiltonian is easily diagonalized with respect to the major index, and we, as might be expected, obtain:

$$\varphi_{11}(\mu;\lambda) = \sqrt{p^2(\mu) + \varkappa_0^2(M)} + [p_\alpha(\mu) - p_\alpha(\lambda)]\,F^{\alpha}_{11}(\mu;\lambda) \qquad (17)$$

where $\varkappa_0^2(M)$ is the constant of integration. From the Hermitian nature of H,

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it follows that $\varkappa_0^2 > 0$, as it should be. The function $F_{11}(\mu;\lambda)$ is determined from (15).

In the general case, (11) has the particular solution:

$$\varphi_{ml} = \exp\left\{ i\sum_{j=1}^{m} g_1(\mu_j) - i\sum_{k=1}^{l} g_1(\lambda_k) + ic_2\left(\sum_{j=1}^{m} p(\mu_j) - \sum_{k=1}^{l} p(\lambda_k)\right)\right\} + \left[\sum_{j=1}^{m} p_\beta(\mu_j) - \sum_{k=1}^{l} p_\beta(\lambda_k)\right] \times \left\{ C_\beta + \left[\sum_{j=1}^{m} p_\alpha(\mu_j) - \sum_{k=1}^{l} p_\alpha(\lambda_k)\right] F_{ml}^{\alpha\beta}(\mu;\lambda)\right\} \quad (m, l \neq 1) \tag{18}$$

$$\varphi_{11}(\mu;\lambda) = \sqrt{p^2(\mu) + \varkappa_0^2(M)}\left\{ 1 + \left[p_\alpha(\mu) - p_\alpha(\lambda)\right]\frac{\partial g_1(\mu)}{\partial p_\alpha(\mu)} + \left(g_2'(0)\right)_\alpha \right\}$$

Here g_1 and g_2 are arbitrary real functions and C_β is an arbitrary constant. The function F_{ml} are determined from (15). For the case of a "triple" Hamiltonian when only φ_{11}, φ_{12}, and φ_{21} are different from zero, a similar solution (with no indication of the method) was reported by Snyder (Hartland S. Snyder, Phys. Rev., 73, 524, 1948). In this paper, the coefficient i in the exponent was left out, apparently because of a typographical error. The general solution of (11) for the case of interacting fields is much more difficult. However, it is not required for the meantime, since the theory is still too general. Actually, up to this time we have not even introduced the concept of coordinates (therefore the tool developed may prove useful for the study of quantized space-time systems) and the interaction can be as "spread out" as desired as well as "point". But the condition of weakening of correlation must be formulated in any theory, and it imposes very important new restrictions on the functions φ_{ml}, to which we hope to return later. We note in conclusion that the considerations developed above remain the same for Fermi fields, except that the calculation is made somewhat more complex because of the antisymmetry of the functions φ_{ml}.........Submitted 11 August 1950

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