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WORKS OF THE CENTRAL SCIENTIFIC RESEARCH INSTITUTE OF GEODESY, AERIAL SURVEYING AND CARTOGRAPHY

Trudy Tsentral'nogo Nauchno-Issledovatel'skogo Instituta Geodesiya, Aeros"emki i Kartografii, No. 121, Moscow, 1957, pp. 1-112.

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TABLES FOR THE CALCULATION OF DEVIATIONS OF VERTICAL LINES ON THE PHYSICAL SURFACE OF THE EARTH AND OF ELEVATIONS OF OF THE QUASI-GEOID

V. F. Yeremeyev

Works of the Central Scientific Research Institute of Geodesy, Aerial Surveying and Cartography, pp 3 - 16

M. S. Molodenskiy obtains, in his article [1], formulas for the elevation ζ of the quasi-geoid and for the deviations ξ and η of the vertical on the physical surface of the Earth:

$$\zeta = \frac{1}{4\pi\gamma_0}\int_\sigma \left[g-\gamma+\frac{3}{2\rho_0}\int_S \frac{\delta\varphi}{r_0}\,dS\right] S(\rho_0,\psi)\,d\sigma + \frac{1}{\gamma_0}\int_S \frac{\delta\varphi}{r_0}\,dS, \quad (1)$$

$$\left.\begin{aligned}\xi &= +\frac{1}{4\pi\gamma_0}\int_\sigma \left[g-\gamma+\frac{3}{2\rho_0}\int_S \frac{\delta\varphi}{r_0}\,dS\right]\frac{\partial S(\rho_0,\psi)}{\rho_0\,\partial\psi}\cos A\,d\sigma - \\ &\quad -\frac{1}{\gamma_0}\int_S \frac{\delta\varphi}{r_0^2}\cos A\,dS + \frac{2\pi}{\gamma_0}\delta\varphi_0\cos(n,x) \\ \eta &= +\frac{1}{4\pi\gamma_0}\int_\sigma \left[g-\gamma+\frac{3}{2\rho_0}\int_S \frac{\delta\varphi}{r_0}\,dS\right]\frac{\partial S(\rho_0,\psi)}{\rho_0\,\partial\psi}\sin A\,d\sigma - \\ &\quad -\frac{1}{\gamma_0}\int_S \frac{\delta\varphi}{r_0^2}\sin A\,dS + \frac{2\pi}{\gamma_0}\delta\varphi_0\cos(n,y)\end{aligned}\right\}, \quad (2)$$

$$2\pi\,\delta\varphi = \delta g + \int \frac{\rho-\rho_0}{r_0^3}\,\delta\varphi\,d\sigma, \quad (3)$$

$$\delta g = -\frac{h}{2\pi}\int \frac{(g-\gamma)-(g-\gamma)_0}{r^3}\,d\sigma. \quad (4)$$

The symbols used here correspond, in the main, to those adopted in the previously cited paper [1] (see drawing).

$S(\rho_0,\psi)$ is the generalized Stokes function;

ρ is the radius vector originating at the center of the reference surface and leading to a variable point S of the surface of the Earth, whose position is approximated by plotting only the normal elevations of points on the surface of the Earth from the reference surface;

ρ_0 represents an identical radius vector for a point S of the surface, corresponding to the point under study of the physical surface;

r_0 is the distance between the points of intersection of radii-vectors ρ and ρ_0 with the reference surface;

r is the distance between the point under investigation situated on the surface S and the intersection of radius vector ρ of the variable point with the reference surface;

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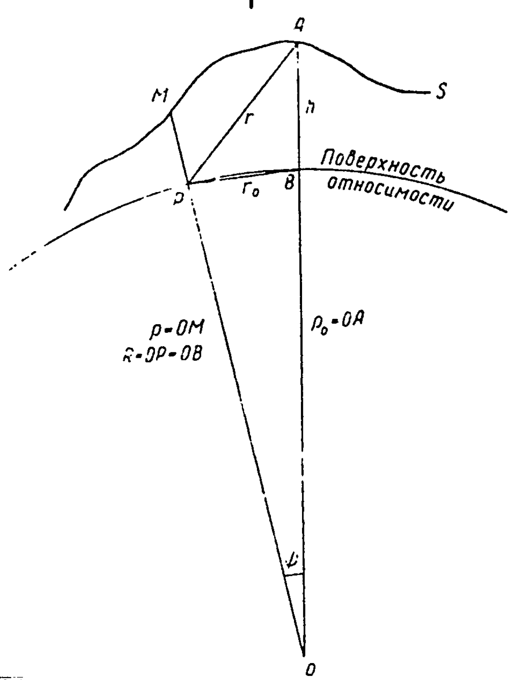
h is the normal elevation of the point under study;
R is the radius of the reference surface;
ψ is the angular distance from the point under study;
$d\sigma$ is the element of the reference surface;
A is the azimuth of element $d\sigma$;
dS is the element of surface S;
n is the orientation of an exterior perpendicular to the surface of the Earth at the point under study;
x and y are the orientations used in calculating deviations ξ and η; x and y pass through the point under study parallel to the reference surface;
g is the measured value for the force of gravity;
γ is the normal value for the force of gravity, corresponding to the measured value; it is calculated for the latitude and normal elevation of the point;
$\delta\varphi$ is the residual density of the surface layer; M. S. Molodenskiy [1] designates this density as $\delta\varphi_1$;
$\delta\varphi_0$ is the value of $\delta\varphi$ at the point under study;



g...
variabl...
(g...
point...

$$S(\rho_0, \psi) = \frac{2}{r} - \frac{3r}{\rho_0^2} + \frac{1}{\rho_0} - \frac{5R}{\rho_0^2}\cos\psi - 3\frac{R}{\rho_0^2}\cos\psi \ln\frac{1}{2\rho_0}(r + \rho_0 - R\cos\psi), \quad (5)$$

$$\frac{\partial S(\rho_0, \psi)}{\rho_0\,\partial\psi} = -2\frac{r_0}{r^3}\cos\frac{\psi}{2} - \frac{3r_0\cos\frac{\psi}{2}}{r\rho_0^2} + 5\frac{R\sin\psi}{\rho_0^3} + 3\frac{R}{\rho_0^3}\sin\psi\ln\frac{1}{2\rho_0}(r + \rho_0 - R\cos\psi) -$$

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$$-\frac{3R}{\rho_0^3}\cos\psi\,\frac{rR\sin\psi+\rho_0 r_0\cos\frac{\psi}{2}}{r\,(r+\rho_0-R\cos\psi)}. \qquad (6)$$

Let us point out that the expression for $\frac{\partial S(\rho_0,\psi)}{\rho_0\,\partial\psi}$ in [3] contains a typographical error. Nevertheless, formula (6) in its correct form is used for calculations in papers [3] and [4]. Table 1 is intended for the calculation of

$$F(\rho_0,\psi)=\frac{R}{2}\,S(\rho_0,\psi)\sin\psi, \qquad (7)$$

while Table makes it possible to obtain

$$Q(\rho_0,\psi)=-\frac{R^2\varkappa''}{2\gamma_0}\,\frac{\partial S(\rho_0,\psi)}{\rho_0\,\partial\psi}\sin\psi. \qquad (8)$$

It has been posited, in compiling the tables (h is to be expressed in kilometers), that:

$$F(\rho_0,\psi)=F(R,\psi)+\Delta F(\rho_0,\psi)=F(R,\psi)+ +h\left[\frac{\partial F(\rho_0,\psi)}{\partial\rho_0}\right]_{\rho_0=R}+\frac{1}{2}h^2\left[\frac{\partial^2 F(\rho_0,\psi)}{\partial\rho_0^2}\right]_{\rho_0=R,} \qquad (9)$$

$$Q(\rho_0,\psi)=Q(R,\psi)+\Delta Q(\rho_0,\psi)=Q(R,\psi)+ +h\left[\frac{\partial Q(\rho_0,\psi)}{\partial\rho_0}\right]_{\rho_0=R}+\frac{1}{2}h^2\left[\frac{\partial^2 Q(\rho_0,\psi)}{\partial\rho_0^2}\right]_{\rho_0=R,} \qquad (10)$$

$$F(R,\psi)=\frac{1}{2}\sin\psi\left[\csc\frac{\psi}{2}-6\sin\frac{\psi}{2}+1-5\cos\psi- -3\cos\psi\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right)\right],$$

$$Q(R,\psi)=\frac{\varkappa''}{2\gamma_0}\cos^2\frac{\psi}{2}\left[\csc\frac{\psi}{2}+12\sin\frac{\psi}{2}-32\sin^2\frac{\psi}{2}+ +\frac{3}{1+\sin\frac{\psi}{2}}-12\sin^2\frac{\psi}{2}\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right)\right].$$

For the function $F(R,\psi)$, tables have been compiled by Lambert [5].

Since the relation $\frac{\rho_0}{R}$ approaches unity ($\frac{h}{R}<\frac{1}{700}$), the corrections $\Delta F(\rho_0,\psi)$ and $\Delta Q(\rho_0,\psi)$ are small in comparison with $F(R,\psi)$ and $Q(R,\psi)$, respectively, when ψ is sufficiently large. Values for $\Delta F(\rho_0,\psi)$ and

$\Delta Q(\rho_o, \psi)$ are more easily interpolated than the functions $F(\rho_o, \psi)$ and $Q(\rho_o, \psi)$. For the major portion of the range of variation of ψ, the values of $\Delta F(\rho_o, \psi)$ and $\Delta Q(\rho_o, \psi)$ are, for practical purposes, directly proportionate to h. Only for small values of ψ, do we have to deal with quadratic values. For $\psi = 0$, we have: $F(R, 0) = +1$, $F(\rho_o\ 0) = 0$, $Q(R, 0) = +\infty$, $Q(\rho_o, 0) = 0$. For this reason, for values of ψ close to zero, there is no sense in introducing functions $\Delta F($ [illegible] Differentiating (5),

$$k_1 = \left(\frac{\partial F(\rho_0, \psi)}{\partial \rho_0}\right)_{\rho_0 = R} = \frac{\cos\frac{\psi}{2}}{2R}\left(-1 + 18\sin^2\frac{\psi}{2} - 2\sin\frac{\psi}{2} + \right.$$

$$\left. + 26\sin\frac{\psi}{2}\cos\psi - 3\cos\psi + 12\sin\frac{\psi}{2}\cos\psi\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right),$$

$$\left(\frac{\partial^2 F(\rho_0, \psi)}{\partial \rho_0^2}\right)_{\rho_0 = R} = \frac{\cos\frac{\psi}{2}}{r_0^2}\left(-1 + 24\sin^2\frac{\psi}{2} - 172\sin^3\frac{\psi}{2} - \right.$$

$$\left. - 144\sin^4\frac{\psi}{2} + 360\sin^5\frac{\psi}{2}\right).$$

The member with h^2 in (9) in small, and differs from zero within the limits of accuracy of the calculations only for $\psi < 8^o.5$. For this reason, it is assumed that:

$$k_2 = -\frac{1}{2}\frac{1}{r_0^2}\cos\frac{\psi}{2}.$$
$$\psi < 8^\circ{,}5$$

The approximate formula

$$F(\rho_0, \psi) = F(R, \psi) + hk_1 + h^2k_2$$

has been checked for the values $\psi = 1, 10, 30, 60$ and 180^o for h = 5 km by comparison with results obtained by the complete formula (5) and (7). The deviation was found to lie within the margin of error in the compilation of Table 1 (below 0.00005). From Δ AOP we get:

$$r^2 = h^2 + 4hR\sin^2\frac{\psi}{2} + 4R^2\sin^2\frac{\psi}{2};$$

Since $r_o = 2R\sin\frac{\psi}{2}$, we get

$$r = r_0\sqrt{1 + \frac{h^2}{r_0^2} + \frac{h}{R}}.$$

The uantity $\frac{h}{R}$ is small by comparison with $\frac{h^2}{r_o^2}$,

and we may therefore posit the approximations

$$\frac{r}{r_0} = \sim\sqrt{1 + \frac{h^2}{r_0^2}} + \frac{1}{\sqrt{1 + \frac{h^2}{r_0^2}}}\frac{h}{2R} \qquad (11)$$

$$\frac{r_0}{r} = \sim\frac{1}{\sqrt{1 + \frac{h^2}{r_0^2}}} - \frac{1}{\left(1 + \frac{h^2}{r_0^2}\right)^{3/2}}\frac{h}{2R}. \qquad (12)$$

Таблица 1

ψ	F (ψ, 0)	$10^4 k_1$	$10^4 k_2$	ψ	F (ψ, 0)	$10^4 k_1$	$10^4 k_2$
1°	+1,0885	−3,4	−0,40	40°	−0,0543	+1,8	
1,1	+1,0945	−3,4	−0,33	41	−0,1061	+2,0	
1,2	+1,1005	−3,4	−0,28	42	−0,1574	+2,2	
1,3	+1,1058	−3,4	−0,24	43	−0,2080	+2,4	
1,4	+1,1111	−3,4	−0,21	44	−0,2579	+2,5	
1,5	+1,1162	−3,4	−0,18	45	−0,3070	+2,7	
1,6	+1,1212	−3,4	−0,16	46	−0,3551	+2,9	
1,7	+1,1259	−3,4	−0,14	47	−0,4023	+3,0	
1,8	+1,1305	−3,4	−0,12	48	−0,4484	+3,2	
1,9	+1,1349	−3,4	−0,11	49	−0,4934	+3,3	
2,0	+1,1392	−3,5	−0,10	50	−0,5371	+3,4	
2,5	+1,1584	−3,5	−0,06	51	−0,5796	+3,6	
3,0	+1,1746	−3,5	−0,04	52	−0,6208	+3,7	
3,5	+1,1883	−3,5	−0,03	53	−0,6606	+3,8	
4,0	+1,1996	−3,5	−0,02	54	−0,6989	+3,9	
4,5	+1,2090	−3,5	−0,02	55	−0,7357	+4,1	
5,0	+1,2165	−3,5	−0,02	56	−0,7710	+4,2	
5,5	+1,2224	−3,5	−0,01	57	−0,8047	+4,3	
6,0	+1,2267	−3,5	−00,1	58	−0,8367	+4,3	
6,5	+1,2294	−3,5	−00,1	59	−0,8671	+4,4	
7,0	+1,2309	−3,5	0	60	−0,8957	+4,5	
7,5	+1,2311	−3,4	0	61	−0,9226	+4,6	
8,0	+1,2300	−3,4	0	62	−0,9477	+4,6	
8,5	+1,2277	−3,4	0	63	−0,9709	+4,7	
9,0	+1,2244	−3,3	0	64	−0,9924	+4,7	
9,5	+1,2200	−3,3	0	65	−1,0120	+4,8	
10,5	+1,2146	−3,3	0	66	−1,0298	+4,8	
11	+1,2008	−3,2	0	67	−1,0457	+4,8	
12	+1,1834	−3,1	0	68	−1,0597	+4,9	
13	+1,1627	−3,0	0	69	−1,0718	+4,9	
14	+1,1388	−2,8	0	70	−1,0820	+4,9	
15	+1,1120	−2,7	0	71	−1,0904	+4,9	
16	+1,0826	−2,6	0	72	−1,0968	+4,9	
17	+1,0506	−2,4	0	73	−1,1014	+4,9	
18	+1,0162	−2,3	0	74	−1,1040	+4,8	
19	+0,9796	−2,1	0	75	−1,1049	+4,8	
20	+0,9410	−1,9	0	76	−1,1039	+4,8	
21	+0,9005	−1,8		77	−1,1011	+4,7	
22	+0,8582	−1,6		78	−1,0965	+4,7	
23	+0,8143	−1,4		79	−1,0902	+4,6	
24	+0,7690	−1,2		80	−1,0821	+4,6	
25	+0,7224	−1,0		81	−1,0723	+4,5	
26	+0,6745	−0,8		82	−1,0609	+4,4	
27	+0,6256	−0,7		83	−1,0478	+4,3	
28	+0,5757	−0,5		84	−1,0331	+4,3	
29	+0,5250	−0,3		85	−1,0169	+4,2	
30	+0,4736	−0,1		86	−0,9992	+4,1	
31	+0,4216	−0,1		87	−0,9800	+4,0	
32	+0,3691	+0,3		88	−0,9594	+3,9	
33	+0,3163	+0,5		89	−0,9374	+3,8	
34	+0,2632	+0,7		90	−0,9142	+3,7	
35	+0,2100	+0,9		91	−0,8897	+3,5	
36	+0,1568	+1,1		92	−0,8640	+3,4	
37	+0,1037	+1,3		93	−0,8372	+3,3	
38	+0,0507	+1,5		94	−0,8093	+3,2	
39	−0,0020	+1,6		95	−0,7804	+3,0	
				96	−0,7506	+2,9	

ψ	F (ψ, 0)	$10^4 k_1$	ψ	F (ψ, 0)	$10^4 k_1$
97°	−0,7199	+2,8	139°	+0,5149	−2,3
98	−0,6883	+2,6	140	+0,5258	−2,3
99	−0,6560	+2,5	141	+0,5353	−2,3
100	−0,6230	+2,3	142	+0,5435	−2,4
101	−0,5894	+2,2	143	+0,5502	−2,4
102	−0,5552	+2,0	144	+0,5556	−2,4
103	−0,5205	+1,9	145	+0,5596	−2,4
104	−0,4854	+1,7	146	+0,5622	−2,4
105	−0,4500	+1,6	147	+0,5634	−2,4
106	−0,4143	+1,4	148	+0,5633	−2,4
107	−0,3784	+1,3	149	+0,5618	−2,4
108	−0,3423	+1,0	150	+0,5590	−2,4
109	−0,3062	+1,0	151	+0,5548	−2,4
110	−0,2700	+0,8	152	+0,5493	−2,3
111	−0,2340	+0,7	153	+0,5425	−2,3
112	−0,1980	+0,5	154	+0,5344	−2,3
113	−0,1622	+0,4	155	+0,5251	−2,2
114	−0,1268	+0,2	156	+0,5146	−2,2
115	−0,0916	+0,1	157	+0,5029	−2,1
116	−0,0568	0	158	+0,4900	−2,1
117	−0,0225	−0,2	159	+0,4759	−2,0
118	+0,0114	−0,3	160	+0,4608	−1,9
119	+0,0226	−0,4	161	+0,4446	−1,9
120	+0,0773	−0,6	162	+0,4274	−1,8
121	+0,1093	−0,7	163	+0,4093	−1,7
122	+0,1405	−0,8	164	+0,3902	−1,6
123	+0,1709	−0,9	165	+0,3702	−1,5
124	+0,2005	−1,1	166	+0,3494	−1,5
125	+0,2292	−1,2	167	+0,3278	−1,4
126	+0,2569	−1,3	168	+0,3055	−1,3
127	+0,2837	−1,4	169	+0,2825	−1,2
128	+0,3094	−1,5	170	+0,2588	−1,1
129	+0,3341	−1,6	171	+0,2346	−1,0
130	+0,3577	−1,7	172	+0,2099	−0,9
131	+0,3801	−1,8	173	+0,1847	−0,8
132	+0,4014	−1,8	174	+0,1591	−0,7
133	+0,4214	−1,9	175	+0,1332	−0,5
134	+0,4402	−2,0	176	+0,1069	−0,4
135	+0,4578	−2,1	177	+0,0804	−0,3
136	+0,4740	−2,1	178	+0,0537	−0,2
137	+0,4890	−2,2	179	+0,0269	−0,1
138	+0,5026	−2,2	180	0	0

Таблица 2

ψ	Q (ψ, 0)	q_1	q_2	ψ	Q (ψ, 0)	q_1	q_2
1°	+12,386	−0,0058	−0,00146	11°	+1,489	−0,0007	
2	+ 6,362	−0,0030	−0,00018	12	+1,402	−0,0007	
3	+ 4,360	−0,0020	−0,00005	13	+1,330	−0,0006	
4	+ 3,362	−0,0016	−0,00001	14	+1,267	−0,0006	
5	+ 2,767	−0,0013	−0,00002	15	+1,213	−0,0006	
6	+ 2,372	−0,0011	−0,00001	16	+1,166	−0,0006	
7	+ 2,092	−0,0010	0	17	+1,124	−0,0005	
8	+ 1,883	−0,0009	0	18	+1,087	−0,0005	
9	+ 1,721	−0,0008	0	19	+1,053	−0,0005	
10	+ 1,593	−0,0008	0	20	+1,022	−0,0005	

ψ	$Q(\psi, 0)$	q_1	ψ	$Q(\psi, 0)$	q_1
21°	+0,993	0,0005	85°	—0,224	—0,0001
22	+0,967	—0,0004	86	—0,237	+0,0002
23	+0,942	0,0004	87	—0,251	+0,0002
24	+0,919	—0,0004	88	—0,264	0,0002
25	+0,896	—0,0004	89	—0,276	0,0002
26	+0,875	—0,0004	90	—0,288	0,0002
27	+0,855	—0,0004	91	—0,300	0,0002
28	+0,835	—0,0004	92	—0,310	0,0002
29	+0,816	—0,0004	93	—0,321	+0,0002
30	+0,796	—0,0004	94	—0,331	0,0002
31	+0,778	—0,0004	95	—0,340	0,0002
32	+0,760	—0,0004	96	—0,349	+0,0002
33	+0,741	—0,0003	97	—0,357	[illegible]
34	+0,723	—0,0003	98	—0,365	[illegible]
35	+0,705	—0,0003	99	—0,372	[illegible]
36	+0,687	—0,0003	100	—0,379	0,[illegible]02
37	+0,669	—0,0003	101	—0,385	+0,[illegible]02
38	+0,651	—0,0003	102	—0,390	+0,[illegible]02
39	+0,633	—0,0003	103	—0,396	+0,0002
40	+0,615	—0,0003	104	—0,400	0,0002
41	+0,596	—0,0003	105	—0,404	+0,0002
42	+0,578	0,0003	106	—0,407	+0,0002
43	+0,559	—0,0002	107	—0,410	0,0002
44	+0,541	—0,0002	108	—0,412	0,0002
45	+0,522	—0,0002	109	—0,414	+0,0002
46	+0,503	—0,0002	110	—0,415	+0,0002
47	+0,484	—0,0002	111	—0,416	+0,0002
48	+0,464	—0,0002	112	—0,416	+0,0002
49	+0,445	—0,0002	113	—0,416	+0,0002
50	+0,426	—0,0002	114	—0,414	+0,0002
51	+0,406	—0,0002	115	—0,413	+0,0022
52	+0,386	—0,0002	116	—0,411	+0,0002
53	+0,366	—0,0001	117	—0,409	+0,0002
54	+0,346	—0,0001	118	—0,406	+0,0002
55	+0,326	—0,0001	119	—0,403	+0,0002
56	+0,306	—0,0001	120	—0,399	+0,0002
57	+0,286	—0,0001	121	—0,395	+0,0002
58	+0,266	—0,0001	122	—0,390	+0,0002
59	+0,246	—0,0001	123	—0,385	+0,0002
60	+0,226	—0,0001	124	—0,380	+0,0002
61	+0,206	—0,0001	125	—0,374	+0,0002
62	+0,186	—0,0001	126	—0,368	+0,0002
63	+0,166	0,0000	127	—0,36[illegible]	+0,0002
64	+0,146	0,0000	128	—0,3[illegible]	0,0002
65	+0,126	0,0000	129	—0,[illegible]	0,0002
66	+0,106	0,0000	130	—0,3[illegible]	0,0002
67	+0,087	0,0000	131	—0,333	+0,0002
68	+0,067	0,0000	132	—0,325	+0,0002
69	+0,048	0,0000	133	—0,317	+0,0002
70	+0,029	0,0000	134	—0,309	0,0002
71	+0,010	0,0000	135	—0,300	0,0002
72	—0,009	+0,0001	136	—0,292	0,0002
73	—0,027	+0,0001	137	—0,283	+0,0002
74	—0,045	+0,0001	138	—0,274	+0,0001
75	—0,063	+0,0001	139	—0,264	+0,0001
76	—0,081	+0,0001	140	—0,255	+0,0001
77	—0,098	+0,0001	141	—0,246	+0,0001
78	—0,115	+0,0001	142	—0,236	+0,0001
79	—0,132	+0,0001	143	—0,227	+0,0001
80	—0,148	+0,0001	144	—0,218	+0,0001
81	—0,164	+0,0001	145	—0,208	+0,0001
82	—0,180	+0,0001	146	—0,199	+0,0001
83	—0,195	+0,0001	147	—0,189	+0,0001
84	—0,209	+0,0001	148	—0,180	+0,0001

ψ	$Q(\psi, 0)$	q_1	ψ	$Q(\psi, 0)$	q_1
149°	−0,171	+0,0001	165	−0,044	0,0000
150	−0,161	+0,0001	166	−0,039	0,0000
151	−0,152	+0,0001	167	−0,034	0,0000
152	−0,143	+0,0001	168	−0,029	0,0000
153	−0,134	+0,0001	169	−0,025	0,0000
154	−0,126	+0,0001	170	−0,020	0,0000
155	−0,117	+0,0001	171	−0,017	0,0000
156	−0,109	+0,0001	172	−0,013	0,0000
157	−0,101	+0,0001	173	−0,010	0,0000
158	−0,093	+0,0001	174	−0,007	0,0000
159	−0,085	0,0000	175	−0,005	0,0000
160	−0,078	0,0000	176	−0,003	0,0000
161	−0,070	0,0000	177	−0,002	0,0000
162	−0,064	0,0000	178	−0,001	0,0000
163	−0,057	0,0000	179	0,000	0,0000
164	−0,051	0,0000	180	0,000	0,0000

By using these relations, a simplified formula was obtained for $F(\rho_0, \psi)$, which is adequately accurate for $0 < \psi$ and $h < 10$ km:

$$\underset{0<\psi<1^\circ}{F(\rho_0, \psi)} \approx \frac{1}{\sqrt{1+\frac{h^2}{r_0^2}}} - \frac{3}{2}\frac{r_0^2}{R^2}\sqrt{1+\frac{h^2}{r_0^2}} - \frac{2r_0}{R} - \frac{3}{2}\frac{r_0}{R}\ln\left[\frac{r_0}{2R}\sqrt{1+\frac{h^2}{r_0^2}} + \frac{h}{2R}\right] - \frac{1}{\left(1+\frac{h^2}{r_0^2}\right)^{3/2}}\frac{h}{2R}.$$

To aid the calculations, it is convenient to introduce the auxiliary angle $\omega = \arctan \frac{h}{r_0}$. We obtain

$$\underset{0<\psi<1^\circ}{F(\rho_0, \psi)} = \cos\omega - 6\sin^2\frac{\psi}{2}\sec\omega - 4\sin\frac{\psi}{2} - 3\sin\frac{\psi}{2}\ln\left[\frac{h}{2R}(1+\csc\omega)\right] - \frac{h}{2R}\cos^3\omega.$$

By differentiating (6), we find

$$q_1 = \left[\frac{\partial Q(\rho_0, \psi)}{\partial \rho_0}\right]_{\rho_0 = R} = -\frac{\varkappa''}{2R\gamma_0}\left\{-9\sin^2\psi\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) + 3\csc\frac{\psi}{2} + 9 + 24\sin\frac{\psi}{2} - 108\sin^2\frac{\psi}{2} - 36\sin^3\frac{\psi}{2} + 108\sin^4\frac{\psi}{2}\right\}.$$

For small values of ψ, we may posit:

$$\left[\underset{\psi<10^\circ}{\frac{\partial^2 Q(\rho_0, \psi)}{\partial \rho_0^2}}\right]_{\rho_0 = R} = -\frac{3}{8}\frac{\varkappa''}{\gamma_0 R^2}\frac{\cos^2\frac{\psi}{2}}{\sin^3\frac{\psi}{2}};$$

$$\underset{\psi<10^\circ}{q_2} = -\frac{3}{4}\frac{\varkappa''}{\gamma_0}\frac{\cos^2\frac{\psi}{2}}{r_0^2}\csc\frac{\psi}{2}.$$

The approximate formula

$$Q(\rho_0, \psi) = Q(R, \psi) + hq_1 + h^2 q_2$$

with given values for q_1 has been used in compiling Table 2 within the range $10^\circ \leqslant \psi \leqslant 180^\circ$. Within this interval, the term $h^2 q_2$ is smaller than the range of accuracy of the table. Within the range $1^\circ \leqslant \psi \leqslant 10^\circ$, we may posit

$$\underset{\psi < 10^\circ}{Q(\rho_0, \psi)} - Q(R, \psi) = -\frac{3\varkappa''}{2R\gamma_0}\left\{\csc\frac{\psi}{2} + 3 + 8\sin\frac{\psi}{2}\right\}h - -\frac{3}{4}\frac{\varkappa''}{\gamma_0}\frac{\csc\frac{\psi}{2}}{r_0^2}h^2.$$

The mean value for γ_0, the normal force of gravity, is taken to equal 979.76 mlg, in accordance with the 1901-1909 normal formula of Helmert. The mean radius R of the Earth is taken to be equal to the radius of a sphere equal in volume to the Krasovskiy ellipsoid, i.e. $R = 6371.126$ km. We did not deem it necessary in Table 2 to calculate values for functions $Q(R, \psi)$ and q_1 at intervals smaller than 1° for the range $1^\circ < \psi < 10^\circ$, since in calculating a grid it is more convenient to operate with the Wening-Meinesz approximate formula of the form $\frac{A}{r} + B + Cr$ with a $q_1 h + q_2 h^2$ correction. By using relations (11) and (12) we may obtain an approximate formula for the interval $0^\circ < \psi < 1^\circ$:

$$\underset{0 < \psi < 1^\circ}{Q(\rho, \psi)} = + \frac{\varkappa'' R}{2\gamma_0}\left\{\frac{2}{r_0}\left[\frac{1}{\left(1+\frac{h^2}{r_0^2}\right)^{3/2}} - \frac{3}{\left(1+\frac{h^2}{r_0^2}\right)^{5/2}}\frac{h}{2R}\right] + + \frac{3r_0}{R^2\sqrt{1+\frac{h^2}{r_0^2}}} + \frac{3}{R\sqrt{1+\frac{h^2}{r_0^2}}}\frac{1}{\sqrt{1+\frac{h^2}{r_0}+\frac{h}{r_0}}}\right\}. \quad (13)$$

The maximum deviation (for $\psi = 1^\circ$) of the value for $Q(\rho_0, \psi)$ calculated from this formula from the correct value amounts to 0.01%.

Let us describe now the calculation of a grid for the determination of deviations ξ and η of the vertical, which need to be calculated at closer intervals and more accurately than ζ. To calculate the elevations ζ of the quasi-geoid, it is possible to use the same grid, transferring coefficients for the effect of specific zones.

The most important in this connection are zones from 0 to ~100 km, whose anomalies determine the local component of the deviation of the vertical. Allowances for the effect of this area are the most difficult to make, particularly in mountainous regions.

The accuracy with which numerical integration is performed in the 0 - 5 km zone is to be selected with reference to the nature of the gravitational field*.

1. For an even gravitational field, such as is usually observed in flat regions, we may posit for the 0-5 km zone:

* See [2] and the paper by the author in the present volume concerning the calculation of the grid.

$$g-\gamma=\Delta g_0+\left(\frac{\partial \Delta g}{\partial r_0}\right) r_0,$$

$$Q(\rho_0, \varphi)=\frac{\varkappa'' R}{\gamma_0} \frac{r_0^2}{\sqrt{(r_0^2+h^2)^3}},$$

where Δg_0 is the anomaly at the point under study. We may assume that the gradient $\frac{\partial \Delta g}{\partial r_0}$ is constant along the radius. Then

$$\Delta\xi=-\frac{\varkappa''}{2\pi\gamma_0}\int_0^{r_1=5\ \kappa\text{м}}\int_0^{2\pi}(g-\gamma)\frac{r_0^2\, dr_0 \cos A\, dA}{\sqrt{(h^2+r_0^2)^3}} \tag{14}$$

or, if in integrating with respect to A we use the formula with eight-ordinate trapezi:

$$\Delta\xi=-\frac{\varkappa''}{8\gamma_0 r_1}\left[\sqrt{h^2+r_1^2}+\frac{h^2}{\sqrt{h^2+r_1^2}}-2h\right]\left[\sum_{n=1}^{8}\Delta g_n \cos A_n\right]_{r_0=r_1}.$$

In final form

$$\Delta\xi=-c(h)\left[\sum_{n=1}^{8}\Delta g_n \cos A_n\right]_{r_0=r_1}, \tag{15}$$

where
$$c(h)=\frac{\varkappa''}{8\gamma_0 r_1}\left[\sqrt{h^2+r_1^2}+\frac{h^2}{\sqrt{h^2+r_1^2}}-2h\right].$$

It is convenient to introduce the auxiliary angle on the basis of the relation.

$$\frac{r_1}{h}=\operatorname{tg}\omega;$$

We then get

$$c(h)=\frac{\varkappa''}{8\gamma_0}\operatorname{ctg}\omega(\cos\omega+\sec\omega-2)$$

For h = 0 we have $c(0)=\frac{\varkappa''}{8\gamma_0}$. The c(h) coefficients are given in Table 3.

Table 3

h м	c (h)	h м	c (h)
0	0",02 632	2000	0",01120
100	0 ,02 528	2200	0 ,01026
200	0 ,02 427	2400	0 ,00939
300	0 ,02 330	2600	0 ,00861
400	0 ,02 236	2800	0 ,00789
500	0 ,02 145	3000	0 ,00723
600	0 ,02 056	3200	[illegible]
700	0 ,01 971	3400	0 ,00610
800	0 ,01 889	3600	0 ,00560
900	0 ,01 810	3800	0 ,00515
1000	0 ,01 734	4000	0 ,00475
1200	0 ,01 591	4200	0 ,00438
1400	0 ,01 458	4400	0 ,00404
1600	0 ,01 335	4600	0 ,00373
1800	0 ,01 223	4800	0 ,00345
		5000	[illegible]

II. In the case of a complex gravitational field, it is possible to use the Gauss formula. In (14), we may posit

$$\int_0^{2\pi} (g-\gamma)\cos A\, dA = f(r_0)$$

and write

$$\Delta\xi = -\frac{\varkappa''}{2\pi\gamma_0}\int_0^{r_1} f(r_0)\frac{r_0^2}{\sqrt{(h^2+r_0^2)^3}}\,dr_0. \qquad (16)$$

We may calculate the integral of (16) by means of the Gauss numerical integration formula with three ordinates:

$$\Delta\xi = -\frac{\varkappa''}{2\pi\gamma_0} r_1\left\{A_1 f(r'_0)\frac{r_0'^2}{\sqrt{(h^2+r_0'^2)^3}} + A_2 f(r_0'')\frac{r_0''^2}{\sqrt{(h^2+r_0''^2)^3}} + A_3 f(r_0''')\frac{r_0'''^2}{\sqrt{(h^2+r_0'''^2)^3}}\right\},$$

where:

$r'_0 = r_1x_1 = 0{,}5635$ км	$x_1 = 0{,}1127$	$A_1 = 0{,}2778$
$r''_0 = r_1x_2 = 2{,}5$ км	$x_2 = 0{,}5$	$A_2 = 0{,}4444$
$r'''_0 = r_1x_3 = 4{,}4365$ км	$x_3 = 0{,}8873$	$A_3 = 0{,}2778$

For calculating the integral of $f(r_0)$, we may carry out numerical integration by means of the trapezium method with eight ordinates. By breaking down the interval 2π into eight equal segments, we may write the approximation:

$$f(r_0) = \frac{\pi}{4}\sum_{n=1}^{8}(g-\gamma)_n \cos A_n.$$

Thus

$$\Delta\xi_{0-5\text{ км}} = -c_1(h)\left[\sum_{n=1}^{8}(g-\gamma)_n\cos A_n\right]_{r_0=r'_0} - c_2(h)\left[\sum_{n=1}^{8}(g-\gamma)_n\cos A_n\right]_{r_0=r''_0} - c_3(h)\left[\sum_{n=1}^{8}(g-\gamma)_n\cos A_n\right]_{r_0=r_0'''}, \qquad (17)$$

where we posit that

$$c_n(h) = \frac{\varkappa''}{8\gamma_0} r_1 A_n \frac{r_0^{n2}}{\sqrt{(r_0^{n2}+h^2)^3}}. \qquad (18)$$

The coefficients for $c_n(h)$ are given in Table 4. Elevation h enters into expressions (15) and (18) as a parameter. By substituting $\sin A_n$ for $\cos A_n$, we obtain the formula for the effect $\Delta\eta$ of the 0-5 km zone on the component of the deviation of the vertical in the plane of the first vertical. From formulas (15) and (18) and Tables 3 and 4 we may establish the dependence of the effect of the 0 - 5 km zone on $\Delta\xi$ and $\Delta\eta$ from elevation h, given a constant gravitational field in the central zone.

Let us calculate the effect of the curvilinear trapezium bounded by arcs of circles with radii r'_0 and r''_0 and radii, whose azimuths equal A_1 and A_2. We may use the integral

$$\int_{r_0'}^{r_0''} Q(\rho, \psi)\, dr_0 = \frac{\varkappa'' R}{\gamma_0}\left\{-\frac{r_0}{\sqrt{r_0^2+h^2}} + \ln\left(r_0+\sqrt{r_0^2+h^2}\right) + \right.$$

$$\left. + \frac{h}{2R}\left[\frac{3r_0}{\sqrt{r_0^2+h^2}} + \frac{r_0^3}{\sqrt{(r_0^2+h^2)^3}} + \frac{3r_0}{h} - 6\ln\left(r_0+\sqrt{r_0^2+h^2}\right)\right]\right\}\Bigg|_{r'_0}^{r''_0}.$$

The effect of the entire 5 - 100 km zone on component ξ will e_ual

$$\Delta\xi_{5-100\,\text{км}} = \sum (g-\gamma)_n c_n,$$

where

$$c = \frac{\varkappa''}{\gamma k}\cos A_m\left\{\frac{r_0}{\sqrt{r_0^2+h^2}} - \ln\left(r_0+\sqrt{r_0^2+h^2}\right) - \frac{h}{2R}\left[\frac{3r_0}{\sqrt{r_0^2+h^2}} + \right.\right.$$

$$\left.\left. + \frac{r_0^3}{\sqrt{(h^2+r_0^2)^3}} + \frac{3r_0}{h} - 6\ln\left(r_0+\sqrt{r_0^2+h^2}\right)\right]\right\}\Bigg|_{r'_0}^{r''_0}.$$

Table 4

h, м	$c_1(h)$, $r'_0 = 563{,}5$ м	$c_2(h)$, $r''_0 = 2500{,}0$ м	$c_3(h)$, $r'''_0 = 4436{,}5$ м
0	0,06487	0,02339	0,00824
100	0,06192	0,02333	0,00823
200	0,05429	0,02317	0,00821
300	0,04461	0,02289	0,00818
400	0,03517	0,02252	0,00814
500	0,02715	0,02205	0,00808
600	0,02081	0,02150	0,00802
700	0,01599	0,02088	0,00794
800	0,01239	0,02021	0,00785
900	0,00969	0,01948	0,00776
1000	0,00768	0,01872	0,00765
1200	0,00498	0,01714	0,00741
1400	0,00338	0,01554	0,00714
1600	0,00238	0,01398	0,00686
1800	0,00173	0,01250	0,00656
2000	0,00129	0,01114	0,00624
2200	0,00099	0,00990	0,00592
2400	0,00078	0,00878	0,00561
2600	0,00062	0,00799	0,00529
2800	0,00050	0,00691	0,00498
3000	0,00041	0,00614	0,00468
3200	0,00034	0,00546	0,00440
3400	0,00028	0,00486	0,00412
3600	0,00024	0,00434	0,00386
3800	0,00020	0,00388	0,00361
4000	0,00018	0,00348	0,00338
4200	0,00015	0,00313	0,00316
4400	0,00013	0,00282	0,00295
4600	0,00012	0,00255	0,00276
4800	0,00010	0,00230	0,00258
5000	0,00009	0,00209	0,00241

Table 5

Values for $\frac{c}{\cos A_m}$, given in units of the fifth decimal figure

No of zones — Radii of — h km

№ зон	Радиусы зон, км \ h км	0	0,5	1,0	1,5	2,0	2,5	3,0	3,5	4,0	4,5	5,0	5,5	6,0	6,5	7,0	7,5
1	5,0 – 7,3	500	495	480	456	426	393	358	323	289	257	228	202	179	158	140	124
2	7,3 – 10,7	500	497	490	488	463	444	423	400	377	352	328	305	282	261	240	221
3	10,7 – 15,7	500	499	495	490	482	472	461	449	434	425	404	388	372	355	339	322
4	15,7 – 22,8	500	499	498	495	491	487	481	475	467	459	450	441	431	421	411	400
5	22,8 – 33,3	500	500	499	498	496	493	491	488	484	480	475	471	465	460	454	448
6	33,3 – 48,5	500	500	499	499	498	497	495	494	492	490	488	485	483	480	477	474
7	48,5 – 70,6	500	500	500	499	499	498	498	497	496	495	494	493	491	490	488	487
8	70,6 – 102,6	500	500	500	499	499	499	499	498	498	497	497	496	495	495	494	493

A_m being the mean azimuth of the trapezium. Values for $\frac{c}{\cos A_m}$ are given in Taole 5.

In allowing for the effect of the area situated beyond a radius $R\psi_1$ = 102.6 km, it is most always possible to use the Wening-Meines formula and the grid, described in [2] and below.

The technique of calculation from formulas (1)-(4) is described in articles [3] and [4]. Let us complete that information. If we posit

$$\frac{1}{2\pi}\int [(g-\gamma)-(g-\gamma)_0]\, dA = \Delta g(r),$$

we obtain

$$\delta g = -h\int \Delta g(r)\frac{r_0\, dr_0}{r^3} \approx -h\Sigma\Delta g_m(r)\int \frac{r_0\, dr_0}{r^3},$$

where Δg_m is the mean anomaly in the zone. By taking into account that

$$r^2 = h^2 + r_0^2 + 2r_0 h \sin\frac{\psi}{2} = h^2 + \frac{\rho^0}{R}r_0^2,$$

we find

$$\int \frac{r_0\, dr_0}{r^3} = -\frac{R}{\rho_0 r}$$

and

$$\delta g = \frac{Rh}{\rho_0}\Sigma\Delta g_m\left[\frac{1}{\sqrt{h^2+\frac{\rho_0}{R}r_0^2}}\right]\Bigg|_{r_0'}^{r_0''} \approx h\Sigma\Delta g_m\left[\frac{1}{\sqrt{h^2+r_0^2}}\right]\Bigg|_{r_0'}^{r_0''}, \qquad (19)$$

(19)

where r'_0 and r''_0 are the inner and outer radii of the ring. It is convenient to intraduce an auxiliary angle, by positing $\frac{h}{r'_0}\sqrt{\frac{R}{\rho_0}} = \tan\omega_1$, or, approximately, $\frac{h}{r_0} = \tan\omega_1$. Then

$$\delta g = \frac{R}{\rho_0}\Sigma\Delta g_m(\sin\omega_2 - \sin_1) \approx \Sigma\Delta g_m(\sin\omega_2 - \sin\omega_1). \qquad (20)$$

(20)

Table 6

No of zones

№ зон	c	r_0 м	№ зон	c	r_0 м
1	—0,1	0	11	—0,0333	7433,7
2	—0,1	484,4	12	—0,0250	9179,2
3	—0,1	750,0	13	—0,0200	11289,2
4	—0,1	1020,1	14	—0,0167	13924,6
5	—0,1	1333,3	15	—0,0143	17387,9
6	—0,1	1732,1	16	—0,0125	22261,2
7	—0,1	2291,1	17	—0,0111	29624,5
8	—0,05	3197,7	18	—0,0100	42140,7
9	—0,05	3873,3	19	—0,0091	67966,1
10	—0,0333	4900,0	20	—0,0083	162006,7
		5913,8			

It is apparent from formula (20) that the effect of individual zones of equal width will drop rapidly with distance away from the point under study, the trend being more marked to the extent that elevation h is smaller. Given zones of equal effect, their width will increase rapidly with distance away from the point under investigation, to the extent that it will be difficult to find mean anomaly in the more distant zones. In testing M. S. Molodenskiy's formulae on a model [3], use was made of the coefficients for $c = (\sin\omega_2 - \sin\omega_1)$ given in Table 6. The same table gives radii r_0 corresponding to h = 1 km. Zonal radii vary in direct proportion to h for the given values of the coefficients.

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ALLOWING FOR THE EFFECT OF DISTANT ZONES ON THE ELEVATION OF THE QUASI-GEOID AND THE DEVIATION OF THE VERTICAL

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The manner of accounting for the effect of distant zones on the elevation ζ of the quasi-geoid has been described by M. S. Molodenskiy (Trudy of the Central Scientific Research Institute of Geodesy, Aerial Surveying and Cartography, No 42, 1945). This paper gives the following formula for ζ:

$$\zeta=\frac{R}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi}\Delta g\,[S(\psi)]-S_m(\psi)]\sin\psi\,d\psi\,d\alpha+\frac{R}{2\gamma}\sum_{n=2}^{m}\Delta g_n K_n(S_m), \qquad (1)$$

where R is the mean radius of the Earth, γ is the mean value of the force of gravity, Δg_n is a spherical function of the n-th order for the resolution of anomalies Δg of the force of gravity, and $S(\psi)$ is the Stokes function:

$$S(\psi)=\csc\frac{\psi}{2}-3\cos\psi\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right)-6\sin\frac{\psi}{2}+1-5\cos\psi,$$

ψ is angular distance from the point concerned, $S_m(\psi)$ is the resolution by spherical functions to the m-th order of the function $S(\psi)$ in the interval $\pi-\psi_0$:

$$S_m(\psi)=\sum_{n=0}^{m}\frac{2n+1}{2}K_n(S_m)\,P_n(\cos\psi),$$

$$K_n(S_m)=\sum_{r=0}^{m}\frac{2r+1}{2}K_r(S)K_n(P_r).$$

Expressions for $K_r(S)$ to $r = 7$ have been developed by M. S. Molodenskiy. In addition, a formula has been obtained for $K_8(S)$. In the formulae that follow, it has been posited that $k = \cos^2\frac{\psi_0}{2}$, $\sqrt{1-k} = t$.

The verification method indicated by M. S. Molodenskiy has been applied for checking the expression for $K_8(S)$

$$K_0(S)k=-4t+5t^2+6t^3-7t^4+(6t^2-6t^4)\ln(t+t^2),$$

$$K_1(S)k^2=-2t+4t^2+\frac{16}{3}t^3-9t^4-2t^5+\frac{11}{3}t^6+2\ln t-$$
$$-2(1-t^2)^3\ln(t+t^2),$$

$$K_2(S)k^3=2-4t+5t^2+4t^3-\frac{19}{2}t^4+3t^6-\frac{1}{2}t^8+6t^2\ln t,$$

$$K_3(S)k^4=1-4t+15t^2-8t^3-4t^4+\frac{28}{5}t^5-\frac{13}{2}t^6+t^8-$$
$$-\frac{1}{10}t^{10}+(6t^2+12t^4)\ln t,$$

$$K_4(S)\,k^5 = \frac{2}{3} - 4t + 22t^2 - \frac{76}{3}t^3 + 35t^4 - 12t^5 - \frac{37}{2}t^6 + \frac{36}{5}t^7 - \frac{11}{2}t^8 + \frac{1}{2}t^{10} - \frac{1}{30}t^{12} + (6t^2 + 30t^4 + 20t^6)\ln t,$$

$$K_5(S)\,k^6 = \frac{1}{2} - 4t + 28t^2 - 48t^3 + \frac{231}{2}t^4 - 88t^5 + \frac{93}{2}t^6 - \frac{528}{35}t^7 - \frac{79}{2}t^8 + \frac{44}{5}t^9 - 5t^{10} + \frac{3}{10}t^{12} - \frac{1}{70}t^{14} + (6t^2 + 54t^4 + 90t^6 + 30t^8)\ln t,$$

$$K_6(S)\,k^7 = \frac{2}{5} - 4t + \frac{67}{2}t^2 - 76t^3 + \frac{493}{2}t^4 - \frac{1352}{5}t^5 + 357t^6 - \frac{1144}{5}t^7 + \frac{79}{4}t^8 - \frac{572}{35}t^9 - \frac{135}{2}t^{10} + \frac{52}{5}t^{11} - \frac{47}{10}t^{12} + \frac{1}{5}t^{14} - \frac{1}{140}t^{16} + (6t^2 + 84t^4 + 252t^6 + 210t^8 + 42t^{10})\ln t$$

$$K_7(S)\,k^8 = \frac{1}{3} - 4t + \frac{387}{10}t^2 - \frac{328}{3}t^3 + 437t^4 - 620t^5 + \frac{2391}{2}t^6 - 1040t^7 + 802t^8 - \frac{31\,460}{63}t^9 - \frac{363}{4}t^{10} - \frac{104}{7}t^{11} - \frac{1543}{15}t^{12} + 12t^{13} - \frac{9}{2}t^{14} + \frac{1}{7}t^{16} - \frac{1}{252}t^{18} + (6t^2 + 120t^4 + 560t^6 + 840t^8 + 420t^{10} + 56t^{12})\ln t,$$

$$K_8(S)\,k^9 = \frac{2}{7} - 4t + \frac{437}{10}t^2 - 148t^3 + \frac{6959}{10}t^4 - \frac{6052}{5}t^5 + \frac{5965}{2}t^6 - \frac{22\,780}{7}t^7 + \frac{8217}{2}t^8 - \frac{22\,100}{7}t^9 + \frac{28\,097}{20}t^{10} - \frac{20\,332}{21}t^{11} - \frac{6949}{20}t^{12} - \frac{68}{7}t^{13} - \frac{1459}{10}t^{14} + \frac{68}{5}t^{15} - \frac{61}{14}t^{16} + \frac{3}{28}t^{18} - \frac{1}{420}t^{20} + (6t^2 + 162t^4 + 1080t^6 + 2520t^8 + 2268t^{10} + 756t^{12} + 72t^{14})\ln t.$$

We may note that, beginning with r = 2, the free term in these expressions equals $\frac{2}{r-1}$, and the t coefficient equals -4; in the ln t polynomial, the coefficient of the first term equals 6, while that of the last one equals $r(r+1)$.

For $r > n$, $K_n(P_r)$ is defined by the formula:

$$\frac{1}{2}K_n(P_r)\,k^r = t^2 + \frac{1}{2}t^4 pq + \frac{1}{3}t^6\,\frac{p(p-1)}{2!}\cdot\frac{q(q-1)}{2!} + \frac{1}{4}t^8\,\frac{p(p-1)(p-2)}{3!}\cdot\frac{q(q-1)(q-2)}{3!} + \ldots,$$

where $p = r - n$, and $q = n + r$. For $r = n$

$$K_r(P_r)\,k^r = \frac{2}{2r+1},$$

and for $r < n$

$$K_n(P_r) = 0.$$

By differentiation of (1), M. S. Molodenskiy has obtained an expression for the ξ and η components of the deviation of the vertical

$$\xi'' = \frac{\rho''}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi}\Delta g\,\frac{d}{d\psi}\left[S(\psi) - S_m(\psi)\right]\sin\psi\cos\alpha\,d\psi\,d\alpha + \frac{\rho''}{2\gamma}\sum_{n=2}^{m}K_n(S_m)\,\frac{\partial\Delta g_n}{\partial\vartheta'}, \qquad (2)$$

$$\eta'' = \frac{\rho''}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi}\Delta g\,\frac{d}{d\psi}\left[S(\psi) - S_m(\psi)\right]\sin\psi\sin\alpha\,\partial\psi\,d\alpha - \frac{\rho''}{2\gamma}\sum_{n=2}^{m}K_n(S_m)\,\frac{\partial\Delta g_n}{\partial L'}\csc\vartheta', \qquad (3)$$

where ϑ' and L' are the polar distance and longitude of the point concerned, and azimuth α is computed from the direction toward North. For $m = \infty$, all of M. S. Molodenskiy's formulae are in strict form.

The effect of the anomalies of distant zones may also be allowed for in the following manner. Let us present the Stokes formula in the form of a sum:

$$\zeta = \frac{R}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi}\Delta g\,S(\psi)\sin\psi\,d\psi\,d\alpha + \frac{R}{4\pi\gamma}\int_{\psi_0}^{\pi}\int_0^{2\pi}\Delta g\,S(\psi)\sin\psi\,d\psi\,d\alpha. \qquad (4)$$

The spherical radius ψ_0 of the allowance for anomalies may be assumed to be large enough to allow the use of anomalies in the second integral in the form of a resolution by spherical functions to the m-th order:

$$\Delta g = \sum_{n=3}^{m}A_{no}P_n(\cos\vartheta) + \sum_{n=2}^{m}\sum_{k=1}^{k=n}A_{nk}\cos kL\,P_{nk}(\cos\vartheta) + \sum_{n=2}^{m}\sum_{k=1}^{k=n}B_{nk}\sin kL\,P_{nk}(\cos\vartheta).$$

Here $P_{nk}(\cos\vartheta)$ are combined Legendre polynomials. It is then possible to perform a double integration in the second integral of (4). Integration with respect to α gives us integrals of the following form:

$$\int_0^{2\pi} P_{nk}(\cos\vartheta)\sin kL\,d\alpha = 2\pi P_{nk}(\cos\vartheta')\sin kL' P_n(\cos\psi),$$

$$\int_0^{2\pi} P_{nk}(\cos\vartheta)\cos kL\,d\alpha = 2\pi P_{nk}(\cos\vartheta')\cos kL' P_n(\cos\psi).$$

To verify these identities, it is possible to use the formula for the regeneration of spherical functions. Both sides must be multiplied by $\frac{2n+1}{4\pi}P_n(\cos\psi)\sin\psi\,d\psi$ and integrated for ψ from 0 to π.

With the aid of these congruences, it is possible to reduce the calculation of the second integral of (4) to the calculation of integrals of the type

$$\int y^n S(y)\,dy. \tag{5}$$

where it is posited that $\cos\psi = y$. Integrals of (5) are, in turn, reduced to tabular ones, while the Stokes formula is reduced to the following expression:

$$\zeta = \frac{P}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi}\Delta g\,S(\psi)\sin\psi\,d\psi\,d\alpha + \frac{R}{2\gamma}\sum_{n=2}^{m}\Delta g_n Q_n, \tag{6}$$

where Q_n are coefficients computed for a given value of ψ_0.

By differentiating (6), we obtain:

$$\xi'' = \frac{\rho''}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi}\Delta g\frac{d}{d\psi}[S(\psi)]\sin\psi\cos\alpha\,d\psi\,d\alpha + \frac{\rho''}{2\gamma}\sum_{n=2}^{m}Q_n\frac{\partial\Delta g_n}{\partial\vartheta'}. \tag{7}$$

$$\eta'' = \frac{\rho''}{4\pi\gamma}\int_0^{\psi_0}\int_0^{2\pi}\Delta g\frac{d}{d\psi}[S(\psi)]\sin\psi\sin\alpha\,d\psi\,d\alpha - \frac{\rho''}{2\gamma}\sum_{n=2}^{m}Q_n\frac{\partial\Delta g_n}{\partial L'}\csc\vartheta' \tag{8}$$

For $m = \infty$, formulae (6), (7) and (8) have a strict form. The greater the value of ψ_0, the more rapid the decrease of coefficients $K_n(S_m)$ and Q_n, and the smaller the m required for calculating ζ, ξ and η from formulas (1), (2), (3) and (6), (7), (8). With the derivation of (6), (7) and (8) the need no longer exists to resolve $S(\psi)$ into a series by spherical functions, and we do not have to determine $S_m(\psi)$. Since

$$K_n(S_m) + \int_1^{y_0} P_n(y)\,S_m(y)\,dy = Q_n$$

and

$$\int_0^{\psi_0}\int_0^{2\pi}\left(\sum_{n=2}^{m}\Delta g_n\right)S_m(\psi)\sin\psi\,d\psi\,d\alpha - \int_0^{\psi_0}\int_0^{2\pi}\Delta g\,S_m(\psi)\sin\psi\,d\psi\,d\alpha \approx 0,$$

sufficiently high values for m allow the switch from formulas (1), (2) and (3) to formulas (6), (7) and (8).

For the calculation of Q_n, it is convenient to put the Stokes function in following form:

$$S(\psi)=S(v)=\frac{1}{v}-3\ln v\,(1+v)+6v^2\ln v\,(1+v)-6v-4+10v^2,$$

where $\sin\frac{\psi}{2} = v$, i.e. $y = 1 - 2v^2$. With the aid of the inte

$$\int\frac{v^n\,dv}{(1+v)}=(-1)^n\ln(1+v)+(-1)^{n-1}v+(-1)^{n-2}\frac{v^2}{2}+$$
$$+(-1)^{n-3}\frac{v^3}{3}+\ldots+\frac{v^n}{n}$$

we find the following relations:

$$\int_1^t vS(v)\,dv=t-\frac{5}{4}t^2-\frac{3}{2}t^3+\frac{7}{4}t^4+\left(-\frac{3}{2}t^2+\frac{3}{2}t^4\right)\ln t(1+t),$$

$$\int_1^t v^3S(v)\,dv=\frac{1}{4}t-\frac{1}{8}t^2+\frac{5}{12}t^3-\frac{7}{8}t^4-t^5+\frac{4}{3}t^6+$$
$$+\left(-\frac{3}{4}t^4+t^6\right)\ln t(1+t)-\frac{1}{4}\ln(1+t),$$

$$\int_1^t v^5S(v)\,dv=\frac{1}{4}t-\frac{1}{8}t^2+\frac{1}{12}t^3-\frac{1}{16}t^4+\frac{1}{4}t^5-\frac{5}{8}t^6-\frac{3}{4}t^7+$$
$$+\frac{17}{16}t^8+\left(-\frac{1}{2}t^6+\frac{3}{4}t^8\right)\ln t(1+t)-\frac{1}{4}\ln(1+t)-\frac{1}{12},$$

$$\int_1^t v^7S(v)\,dv=\frac{9}{40}t-\frac{9}{80}t^2+\frac{3}{40}t^3-\frac{9}{160}t^4+\frac{9}{200}t^5-\frac{3}{80}t^6+\frac{7}{40}t^7-$$
$$-\frac{77}{160}t^8-\frac{3}{5}t^9+\frac{22}{25}t^{10}+\left(-\frac{3}{8}t^8+\frac{3}{5}t^{10}\right)\ln t(1+t)-$$
$$-\frac{9}{40}\ln(1+t)-\frac{9}{80},$$

$$\int_1^t v^9S(v)\,dv=\frac{1}{5}t-\frac{1}{10}t^2+\frac{1}{15}t^3-\frac{1}{20}t^4+\frac{1}{25}t^5-\frac{1}{30}t^6+\frac{1}{35}t^7-$$
$$-\frac{1}{40}t^8+\frac{4}{30}t^9-\frac{39}{100}t^{10}-\frac{1}{2}t^{11}+\frac{3}{4}t^{12}+\left(-\frac{3}{10}t^{10}+\right.$$
$$\left.+\frac{1}{2}t^{12}\right)\ln t(1+t)-\frac{1}{5}\ln(1+t)-\frac{101}{840},$$

$$\int_1^t v^{11}S(v)\,dv=\frac{5}{28}t-\frac{5}{56}t^2+\frac{5}{84}t^3-\frac{5}{112}t^4+\frac{1}{28}t^5-\frac{5}{168}t^6+$$

$$+\frac{5}{196}t^7 \quad \frac{5}{224}t^8+\frac{5}{252}t^9-\frac{1}{56}t^{10}+\frac{3}{28}t^{11}-\frac{55}{168}t^{12}-$$
$$-\frac{3}{7}t^{13}+\frac{32}{49}t^{14}+\left(-\frac{1}{4}t^{12}+\frac{3}{7}t^{14}\right)\ln t\,(1+t)-$$
$$-\frac{5}{28}\ln(1+t)-\frac{241}{2016},$$

$$\int_1^t v^{13}S(v)\,dv=\frac{9}{56}t-\frac{9}{112}t^2+\frac{3}{56}t^3-\frac{9}{224}t^4+\frac{9}{280}t^5-\frac{3}{112}t^6+$$
$$+\frac{9}{392}t^7-\frac{9}{448}t^8+\frac{1}{56}t^9-\frac{9}{560}t^{10}+\frac{9}{616}t^{11}-\frac{9}{672}t^{12}+$$
$$+\frac{5}{56}t^{13}-\frac{221}{784}t^{14}-\frac{3}{8}t^{15}+\frac{37}{64}t^{16}+\left(-\frac{3}{14}t^{14}+\right.$$
$$\left.+\frac{3}{8}t^{16}\right)\ln t\,(1+t)-\frac{9}{56}\ln(1+t)-\frac{1423}{12\,320},$$

$$\int_1^t v^{15}S(v)\,dv=\frac{7}{48}t-\frac{7}{96}t^2+\frac{7}{144}t^3-\frac{7}{192}t^4+\frac{7}{240}t^5-\frac{7}{288}t^6+$$
$$+\frac{7}{336}t^7-\frac{7}{384}t^8+\frac{7}{432}t^9-\frac{7}{480}t^{10}+\frac{7}{528}t^{11}-\frac{7}{576}t^{12}+$$
$$+\frac{7}{624}t^{13}-\frac{7}{672}t^{14}+\frac{11}{144}t^{15}-\frac{95}{384}t^{16}-\frac{1}{3}t^{17}+\frac{14}{27}t^{18}+$$
$$+\left(-\frac{3}{16}t^{16}+\frac{1}{3}t^{18}\right)\ln t\,(1+t)-\frac{7}{48}\ln(1+t)-\frac{45\,401}{411\,840},$$

$$\int_1^t v^{17}S(v)\,dv=\frac{2}{15}t-\frac{1}{15}t^2+\frac{2}{45}t^3-\frac{1}{30}t^4+\frac{2}{75}t^5-\frac{1}{45}t^6+\frac{2}{105}t^7-$$
$$-\frac{1}{60}t^8+\frac{2}{135}t^9-\frac{1}{75}t^{10}+\frac{2}{165}t^{11}-\frac{1}{90}t^{12}+\frac{2}{195}t^{13}-$$
$$-\frac{1}{105}t^{14}+\frac{2}{225}t^{15}-\frac{1}{120}t^{16}+\frac{1}{15}t^{17}-\frac{119}{540}t^{18}-\frac{3}{10}t^{19}+$$
$$+\frac{47}{100}t^{20}+\left(-\frac{1}{6}t^{18}+\frac{3}{10}t^{20}\right)\ln t\,(1+t)-$$
$$-\frac{2}{15}\ln(1+t)-\frac{188\,611}{1\,801\,800}.$$

The Legendre polynomials may be expressed in terms of V:

$$P_2=1-6v^2+6v^4,$$
$$P_3=1-12v^2+30v^4-20v^6,$$
$$P_4=1-20v^2+90v^4-140v^6+70v^8,$$
$$P_5=1-30v^2+210v^4-560v^6+630v^8-252v^{10},$$
$$P_6=1-42v^2+420v^4-1680v^6+3150v^8-2772v^{10}+924v^{12},$$
$$P_7=1-56v^2+756v^4-4200v^6+11\,550v^8-16\,632v^{10}+12\,012v^{12}-$$
$$-3432v^{14},$$
$$P_8=1-72v^2+1260v^4-9240v^6+34\,650v^8-72\,072v^{10}+84\,084v^{12}-$$
$$-51\,480v^{14}+12\,870v^{16}.$$

By using the expressions for P_n and $\int_1^t v^n S(v)\,dv$, we can find:

$$Q_n = \int_1^{y_0} P_n(y)\,S(y)\,dy = -4\int_1^t P_n(v)\,S(v)\,v\,dv,$$

Thus, we obtain:

$$Q_2 = 2 - 4t + 5t^2 + 14t^3 - \frac{53}{2}t^4 - 30t^5 + 47t^6 + 18t^7 - \frac{51}{2}t^8 + (6t^2 - 24t^4 + 36t^6 - 18t^8)\ln t\,(1+t),$$

$$Q_3 = 1 - 4t + 5t^2 + 22t^3 - 46t^4 - \frac{372}{5}t^5 + 136t^6 + 104t^7 - 166t^8 - 48t^9 + \frac{352}{5}t^{10} + (6t^2 - 42t^4 + 108t^6 - 120t^8 + 48t^{10})\ln t\,(1+t),$$

$$Q_4 = \frac{2}{3} - 4t + 5t^2 + \frac{98}{3}t^3 - 72t^4 - 156t^5 + 320t^6 + 360t^7 - 645t^8 - \frac{1120}{3}t^9 + 602t^{10} + 140t^{11} - 210t^{12} + (6t^2 - 66t^4 - 260t^6 - 480t^8 + 420t^{10} - 140t^{12})\ln t\,(1+t).$$

$$Q_5 = \frac{1}{2} - 4t + 5t^2 + 46t^3 - \frac{209}{2}t^4 - 294t^5 + 655t^6 + \frac{6830}{7}t^7 - 1930t^8 - 1660t^9 + 2936t^{10} + 1368t^{11} - 2220t^{12} - 432t^{13} + \frac{4608}{7}t^{14} + (6t^2 - 96t^4 + 540t^6 - 1470t^8 + 2100t^{10} - 1512t^{12} + 432t^{14})\ln t\,(1+t),$$

$$Q_6 = \frac{2}{5} - 4t + 5t^2 + 62t^3 - \frac{287}{2}t^4 - \frac{2562}{5}t^5 + 1211t^6 + 2274t^7 - \frac{19\,509}{4}t^8 - 5558t^9 + 10\,689t^{10} + 7434t^{11} - \frac{26\,061}{2}t^{12} - 5082t^{13} + 8283t^{14} + 1386t^{15} - \frac{8547}{4}t^{16} + (6t^2 - 132t^4 + 1008t^6 - 3780t^8 + 7812t^{10} - 9072t^{12} + 5544t^{14} - 1386t^{16})\ln t\,(1+t),$$

$$Q_7 = \frac{1}{3} - 4t + 5t^2 + \frac{242}{3}t^3 - 189t^4 - 840t^5 + 2072t^6 + 4768t^7 - 10\,913t^8 - \frac{140\,000}{9}t^9 + 32\,186t^{10} + 29\,708t^{11} - \frac{167\,860}{3}t^{12} - 32\,648t^{13} + 56\,848t^{14} + \frac{57\,200}{3}t^{15} - 31\,174t^{16} - 4576t^{17} + \frac{64\,064}{9}t^{18} + (6t^2 - 174t^4 + 1736t^6 - 8568t^8 + 23\,940t^{10} - 39\,732t^{12} + 38\,808t^{14} - 20\,592t^{16} + 4576t^{18})\ln t\,(1+t),$$

$$Q_8 = \frac{2}{7} - 4t + 5t^2 + 102t^3 - 241t^4 - \frac{6552}{5}t^5 + 3336t^6 + \frac{64\,608}{7}t^7 - 22\,251t^8 - 38\,368t^9 + \frac{422\,598}{5}t^{10} + 97\,380t^{11} - 195\,756t^{12} - 151\,800t^{13} + 281\,424t^{14} + \frac{706\,992}{5}t^{15} - 244\,959t^{16} - 72\,072t^{17} + 118\,118t^{18} + 15\,444t^{19} - \frac{120\,978}{5}t^{20} + (6t^2 - 222t^4 + 2808t^6 - 17\,640t^8 + 63\,756t^{10} - 141\,372t^{12} + 195\,624t^{14} - 164\,736t^{16} + 77\,220t^{18} - 15\,444t^{20})\ln t(1+t).$$

For checking the coefficients of the expressions for Q_n, we may use the equation:

$$\frac{dQ_n}{dt} = -4P_n(t)S(t)t.$$

In addition, when $\varphi_0 = 180^\circ$, $Q_n = 0$. In that case, $t = 1$ and $\ln t(1+t) = \ln 2$. Therefore, the sum of the coefficients of the polynomial for $\ln t(1+t)$ must equal zero. The sum of all the other coefficients also equals zero.

For $t = 0$, $Q_n = \frac{2}{n-1}$, and therefore the free term of the expression for Q_n equals $\frac{2}{n-1}$. Let us note that the coefficient for t equals -4, while the coefficient for t^2 equals $+5$.

We give the first values for Q_n

$R\psi_0$ к.м \ Q_n	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8
1000	+1,634	+0.639	+0.311	+0.151	+0,060	+0,003	−0,033
1500	+1,457	+0.472	+0,159				
2000	+1,305	+0,338	+0,046	−0,072	−0,118	−0,128	−0 118

M. S. MOLODENSKIY'S ELLIPTICAL GRID FOR THE CALCULATION OF OF ELEVATIONS OF THE QUASI-GEOID

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The computation of an elliptical grid for astronomical and gravimetric leveling has been described by M. S. Molodenskiy in a paper which appeared in 1937. The object of the present paper is to test the applicability of the elliptical grid of Molodenskiy in astronomical and gravimetric leveling of high accuracy, based on gravimetric cluster surveys.

In the Molodenskiy grid, the effect of the immediate surroundings of astronomical points contained in quadrangles whose sides measure 0.25 l and 0.50 l (where l is the half-distance between points) is determined from gradients of gravitational anomalies. At that time, a similar method was used to account for the effect of the central zone (with a radius of up to 5 km) on gravimetric deviations of the vertical. It was later discovered that the effect of the central zone should not be calculated in simplified form from anomaly gradients when calculating deviations of the vertical even in areas of flat relief, if conclusions are drawn from cluster surveys. If such a procedure is employed, the value of a detailed gravimetric survey is, in some measure, lost. The effect of the central zone on the deviation of the vertical is determined today by means of the Gauss numerical integration formula with three ordinates. It is efficient to calculate the effect of the closer areas in astronomic and gravimetric leveling by means of the Gauss formula only in mountain and foothill areas, when astronomical points are at high altitudes (when the relation k of the elevation of the astronomical point to the half-distance between points is greater than 0.01). In the neighborhood of astronomical points of low elevation or, more exactly, when k < 0.01, the subintegral function varies at a rapid rate, and numerical integration involves considerable errors. This is dealt with in greater detail in [4].

In the present paper, we offer certain clarifications to Molodenskiy's article [1] and correct some of the misprints noted in the latter. For this purpose, the presentation follows as closely as possible that in paragraphs 9 and 10 of that paper [1], and Molodenskiy's symbols and numbering of formulas has been retained.

As Molodenskiy has shown, the difference in elevations of the quasi-geoid in astronomical points is expressed by the following relation:

$$N_B - N_A = \Delta N_0 + \Delta N_g,$$

where ΔN_0 is the result of astronomical leveling, and ΔN_g

is the gravimetric correction for the non-linearity of the variation of the deviations of the vertical. The gravimetric correction is found from the formula

$$\Delta N_g = \int_\Sigma \Delta g L d\Sigma, \qquad (17)$$

where

$$L = \frac{1}{2\pi\gamma}\left[\frac{1}{r_2} - \frac{1}{r_1} - l\left(\frac{x-l}{r_2^3} + \frac{x+l}{r_1^3}\right)\right] \qquad (18)$$

or

$$\Delta N_g = \iint_\Sigma \Delta g M(a, b)\, \delta c\, \delta d, \qquad (26)$$

$$M(a, b) = \frac{1}{2\pi\gamma}\left\{\frac{2}{a} - \frac{a^2-b^2}{ab}\left[\frac{ab-l^2}{(a-b)^3} + \frac{ab+l^2}{(a+b)^3}\right]\right\}, \qquad (27)$$

where γ is the normal value for the force of gravity, and Σ is the area of the astronomical points under study. In astronomic-gravimetric leveling, anomalies of the force of gravity Δg must be known throughout area Σ.

The function M (a, b) is expressed by Molodenskiy in a system of coordinates formed by a system of co-focal ellipses and hyperbolas with foci in projections A and B of the astronomical points on the reference surface, which may be considered plane.

The position of any point of area Σ is defined by quantities a and b, the semi-axes of the ellipse and hyperbola passing through that point. The quantities a and b are related to r_1 and r_2 in the following manner: $2a = r_1 + r_2$, $2b = r_1 - r_2$. Quantities c and d are related to a and b by the equations of the ellipse and the hyperbola:

$$\frac{x^2}{a^2} + \frac{y^2}{c^2} = 1; \quad \frac{x^2}{b^2} - \frac{y^2}{d^2} = 1,$$

for

$$c^2 = a^2 - l^2, \quad x = \frac{ab}{l}, \quad d^2 = l^2 - b^2, \quad y = \frac{cd}{l}.$$

X and y represent rectangular coordinates. The origin of the rectangular coordinate system is chosen at the mid-point of segment AB. The x axis coincides with direction AB. r_1 and r_2 are distances from element $d\Sigma$ to, respectively, points A and B.

Let us break down the integral defining ΔN_g into ΔN_A, ΔN_B and ΔN_C. The first two (ΔN_A and ΔN_B) extend over the surface of the perifocal sections, bounded by the coordinate lines a_o and b_o, while the third (ΔN_C) extends over the remainder of area Σ.

Let us examine the integral extending over the perifocal section which includes B.

From integral (17), we obtain

$$\Delta N_B = \frac{1}{2\pi\gamma}\int \Delta g\left(\frac{1}{r^2}-\frac{1}{r_1}\right)d\Sigma - \frac{l}{2\pi\gamma}\int \Delta g\,\frac{\cos\alpha}{r_2^2}\,d\Sigma - $$
$$- \frac{l}{2\pi\gamma}\int \Delta g\,\frac{\cos\beta}{r_1^2}\,d\Sigma \qquad (28) \quad (28)$$

or

$$\Delta N_B = -\frac{l}{2\pi\gamma}J_1 - \frac{l}{2\pi\gamma}J_2 + \frac{1}{2\pi\gamma}J_3, \qquad (29) \quad (29)$$

where α is the polar angle including pole B and polar axis AB (the angle between r_2 and AB), and β is the polar angle including pole A and polar axis AB (the angle between r_1 and AB);

$$J_1 = \int \Delta g\,\frac{\cos\alpha}{r_2^2}\,d\Sigma, \quad J_2 = \int \Delta g\,\frac{\cos\beta}{r_1^2}\,d\Sigma,$$
$$J_3 = \iint \frac{2\Delta g}{a}\,\delta c\,\delta d = \int \Delta g\left(\frac{1}{r_2}-\frac{1}{r_1}\right)d\Sigma. \qquad (30) \quad 0)$$

In calculating J_1 and J_3 we postulate;

$$\Delta g = \Delta g_B + \left(\frac{\partial g}{\partial x}\right)_B r_2\cos\alpha + \left(\frac{\partial g}{\partial y}\right)_B r_2\sin\alpha. \qquad (30') \quad 0')$$

Then

$$J_1 = \Delta g_B\iint \frac{\cos\alpha}{r_2}\,dr\,d\alpha + \left(\frac{\partial g}{\partial x}\right)_B\iint \cos^2\alpha\,dr\,d\alpha +$$
$$+\left(\frac{\partial g}{\partial y}\right)_B\iint \cos\alpha\sin\alpha\,dr\,d\alpha, \qquad (31) \quad 1)$$

since

$$d\Sigma = r_2\,dr\,d\alpha.$$

The area of integration is symmetrical relative to the x axis, since on the right-hand side of this equation, the first and second integrals amount to double integrals for the lower half of the area, while the third equals zero. Performing this integration for r_2, we obtain:

$$J_1 = 2\Delta g_B\int_0^\pi \ln\frac{r_2}{\rho}\cos\alpha\,d\alpha + 2\left(\frac{\partial g}{\partial x}\right)_B\int_0^\pi r_2\cos^2\alpha\,d\alpha, \qquad (32) \quad (32)$$

where ρ is the radius of any given circumference enclosed within the area of integration. To perform integration with respect to α it is necessary to define r_2 as a function of α. If we posit 1 (letter) = 1 (one), we obtain

$$\cos\alpha = \frac{x-1}{r_2} = \frac{ab-1}{a-b},$$

whence

$$b = \frac{1+a\cos\alpha}{a+\cos\alpha}; \quad a = \frac{b\cos\alpha-1}{\cos\alpha-b}.$$

Therefore, we obtain on line $a = a_0$

$$r_2 = \frac{a_0^2-1}{a_0+\cos\alpha}, \qquad (33) \quad (33)$$

and, on line $b = b_0$:

$$r_2 = \frac{1 - b_0^2}{b_0 - \cos\alpha}. \qquad (34)$$

(34)

The value for α corresponding to the point with curvilinear coordinates a_0 and b_0 will be obtained from:

$$\alpha_0 = \arc\cos\frac{a_0 b_0 - 1}{a_0 - b_0}.$$

Thus, by breaking down each of the integrals on the right-hand side in J_1 into two with limits from 0 to α_0 a[illegible] from (33) and (34), w[illegible]

$$\frac{1}{2} J_1 = \Delta g_B \int_0^{\alpha} \ln\frac{a_0^2 - 1}{\rho(a_0 + \cos\alpha)} \cos\alpha\, d\alpha + $$

$$+ \Delta g_B \int_{\alpha_0}^{\pi} \ln\frac{1 - b_0^2}{\rho(b_0 - \cos\alpha)} \cos\alpha\, d\alpha + \left(\frac{\partial g}{\partial x}\right)_B (a_0^2 - 1) \int_0^{\alpha_0} \frac{\cos^2\alpha}{a_0 + \cos\alpha} d\alpha +$$

$$+ \left(\frac{\partial g}{\partial x}\right)_B (1 - b_0^2) \int_{\alpha_0}^{\pi} \frac{\cos^2\alpha}{b_0 - \cos\alpha} d\alpha. \qquad (35)$$

By taking into account the following equations:

$$\int_0^{\alpha_0} \ln\frac{a_0^2 - 1}{\rho(a_0 + \cos\alpha)} \cos\alpha\, d\alpha = + \left(\ln\frac{r_0}{\rho}\right)\sin\alpha_0 - (a_0\alpha_0 - \sin\alpha_0) +$$

$$+ (a_0^2 - 1)\int_0^{\alpha_0} \frac{d\alpha}{a_0 + \cos\alpha}$$

$$r_0 = \frac{a_0^2 - 1}{a_0 + \cos\alpha_0} = \frac{1 - b_0^2}{b_0 - \cos\alpha_0}$$

$$\int_{\alpha_0}^{\pi} \ln\frac{1 - b_0^2}{\rho(b_0 - \cos\alpha)} \cos\alpha\, d\alpha = -\left(\ln\frac{r_0}{\rho}\right)\sin\alpha_0 + b_0(\pi - \alpha_0) -$$

$$- \sin\alpha_0 + (1 - b_0^2)\int_{\alpha_0}^{\pi} \frac{d\alpha}{b_0 - \cos\alpha}$$

$$\int_0^{\alpha_0} \frac{\cos^2\alpha}{a_0 + \cos\alpha} d\alpha = -a_0\alpha_0 + \sin\alpha_0 + a_0^3 \int_0^{\alpha_0} \frac{d\alpha}{a_0 + \cos\alpha} \qquad (36)$$

$$\int_{\alpha_0}^{\pi} \frac{\cos^2\alpha}{b_0 - \cos\alpha} d\alpha = -b_0(\pi - \alpha_0) + \sin\alpha_0 + b_0^2 \int_{\alpha_0}^{\pi} \frac{d\alpha}{b_0 - \cos\alpha}$$

$$\int_0^{\alpha_0} \frac{d\alpha}{a_0 + \cos\alpha} = \frac{2}{c_0} \operatorname{arc\,tg}\left(\operatorname{tg}\frac{\alpha_0}{2}\,\frac{a_0 - 1}{c_0}\right),$$

$$\int_{\alpha_0}^{\pi} \frac{d\alpha}{b_0 - \cos\alpha} = -\frac{1}{d_0} \ln\frac{d_0 \operatorname{tg}\frac{\alpha_0}{2} - (1 - b_0)}{d_0 \operatorname{tg}\frac{\alpha_0}{2} + (1 - b_0)},$$

$$\operatorname{tg}\frac{\alpha_0}{2} = \frac{(a_0 + 1)(1 - b_0)}{c_0 d_0},$$

$$\operatorname{arc\,tg}\left(\operatorname{tg}\frac{\alpha_0}{2}\cdot\frac{a_0-1}{c_0}\right)=\operatorname{arc\,tg}\frac{1-b_0}{d_0}=\frac{\psi_0}{2}=\frac{1}{2}\arcsin d_0,$$

$$\ln\frac{d_0\operatorname{tg}\frac{\alpha_0}{2}-(1-b_0)}{d_0\operatorname{tg}\frac{\alpha_0}{2}+(1-b_0)}=\ln\frac{1+a_0-c_0}{1+a_0+c_0},$$

we obtain

$$\frac{1}{2}J_1=\Delta g_B\left[b_0(\pi-\alpha_0)-a_0\alpha_0+c_0\psi_0-d_0\ln\frac{1+a_0-c_0}{1+a_0+c_0}\right]+$$
$$+\left(\frac{\partial g}{\partial x}\right)_B\left[c_0^2\left(\sin\alpha_0-\alpha_0 a_0+a_0^2\frac{\psi_0}{c_0}\right)+d_0^2\left(\sin\alpha_0-b_0(\pi-\alpha_0)-\right.\right.$$
$$\left.\left.-\frac{b_0^2}{d_0}\ln\frac{1+a_0-c_0}{1+a_0+c_0}\right)\right]. \qquad (37)$$

(37)

Let us now pass on to J_2. Since in this integral the subintegral function does not tend toward infinity in the area of integration but, on the contrary, assumes small values, it is enough to posit $\Delta g = \Delta g_B$ throughout the section. Proceeding as before, we obtain

$$\frac{1}{2}J_2=\Delta g_B\left(\int_0^{\beta_0}\ln r_1''\cos\beta\,d\beta-\int_0^{\beta_0}\ln r_1'\cos\beta\,d\beta\right), \qquad (38)$$

(38)

where

$$\beta_0=\arccos\left(\frac{x+1}{r_1}\right)=\arccos\frac{a_0b_0+1}{a_0+b_0},$$

$$r_1''=\frac{c_0^2}{a_0-\cos\beta} \qquad (r_1 \text{ is tangential to the ellipse})$$

$$r_1'=\frac{d_0^2}{\cos\beta-b_0} \qquad (r_1 \text{ " to the hyperbola}).$$

Taking into account the equations obtained earlier and the following

$$\int_0^{\beta_0}\frac{d\beta}{a_0-\cos\beta}=\frac{2}{c_0}\operatorname{arc\,tg}\left(\frac{a_0+1}{c_0}\operatorname{tg}\frac{\beta_0}{2}\right),$$

$$\operatorname{tg}\frac{\beta_0}{2}=\frac{(1-b_0)(a_0-1)}{c_0d_0},$$

$$\operatorname{arc\,tg}\left(\operatorname{tg}\frac{\beta_0}{2}\,\frac{a_0+1}{c_0}\right)=\operatorname{arc\,tg}\frac{1-b_0}{d_0}=\frac{\psi_0}{2},$$

$$\int_0^{\beta_0}\frac{d\beta}{b_0-\cos\beta}=\frac{1}{d_0}\ln\frac{1+c_0-a_0}{a_0-1+c_0},$$

we hav

$$\frac{1}{2}J_2=\Delta g_B\left[(a_0-b_0)\beta_0-c_0\psi_0-d_0\ln\frac{c_0-a_0+1}{c_0+a_0-1}\right]. \qquad (39)$$

(39)

We pass on now to J_3. We may substitute (30') into expression (30), allowing for the fact that

$$x = r_2 \cos\alpha + l = r_2 \cos\alpha + 1 = ab,$$

i.e.

$$r_2 \cos\alpha = ab - 1, \quad y = r_2 \sin\alpha = cd.$$

$$J_3 = 2\Delta g_B \int_0^{d_0} \delta d \int_{-c_0}^{+c_0} \frac{\delta c}{\sqrt{1+c^2}} + 2\left(\frac{\partial g}{\partial x}\right)_B \left[\int_0^{d_0} b\,\delta d \int_{-c_0}^{+c_0} \delta c - \int_0^{d_0} \delta d \int_{-c_0}^{+c_0} \frac{1}{a}\,\delta c\right] + 2\left(\frac{\partial g}{\partial y}\right)_B \int_0^{d_0} d\,\delta d \int_{-c_0}^{+c_0} \frac{c}{a}\,\delta c.$$

By performing the integration, we obtain:

$$J_3 = 4\Delta g_B d_0 \ln(a_0 + c_0) + 2\left(\frac{\partial g}{\partial x}\right)_B c_0 (d_0 b_0 + \arcsin d_0) - 4\left(\frac{\partial g}{\partial x}\right)_B d_0 \ln(a_0 + c_0). \qquad (40)$$

$$\operatorname{tg}\frac{\alpha_0 - \beta_0}{2} = \frac{d_0}{c_0} \quad \text{и} \quad \operatorname{tg}\frac{\alpha_0 + \beta_0}{2} = \frac{a_0 d_0}{b_0 c_0},$$

$$J_3 - J_1 - J_2 = 2\Delta g_B \left[-b_0\pi + 2a_0 \operatorname{arctg}\frac{d_0}{c_0} + 2b_0 \operatorname{arctg}\frac{a_0 d_0}{c_0 b_0}\right] + 2\left(\frac{\partial g}{\partial x}\right)_B [-d_0 c_0 (a_0 + b_0) + \alpha_0 (a_0 c_0^2 - b_0 d_0^2) + b_0 d_0^2 \pi - c_0 a_0^2 \psi_0 + b_0^2 d_0 \ln\frac{1 + a_0 - c_0}{1 + a_0 + c_0} + c_0 d_0 b_0 + c_0 \arcsin d_0 - 2d_0 \ln(a_0 + c_0)]$$

or, following some simple transformations:

$$J_3 - J_1 - J_2 = 4\Delta g_B \left[-b_0 \operatorname{arctg}\frac{b_0 c_0}{a_0 d_0} + a_0 \operatorname{arctg}\frac{d_0}{c_0}\right] + 2\left(\frac{\partial g}{\partial x}\right)_B [-c_0 d_0 (a_0 + b_0) + \alpha_0 (a_0 c_0^2 - b_0 d_0^2) + b_0 d_0^2 \pi - c_0 a_0^2 \psi_0 - b_0^2 d_0 \ln(a_0 + c_0) + c_0 d_0 b_0 + c_0 \arcsin d_0 - 2d_0 \ln(a_0 + c_0)]. \qquad (41)$$

Therefore

$$\frac{\Delta N_B}{l} = \frac{1}{2\pi\gamma}(J_3 - J_1 - J_2) = \frac{2\Delta g_B}{\pi\gamma}\left[-b_0 \operatorname{arctg}\frac{b_0 c_0}{a_0 d_0} + a_0 \operatorname{arctg}\frac{d_0}{c_0}\right] + \frac{1}{\pi\gamma}\left(\frac{\partial g}{\partial x}\right)_B [-c_0 d_0 (a_0 + b_0) + \alpha_0 (a_0 c_0^2 - b_0 d_0^2) + b_0 d_0^2 \pi - c_0 a_0^2 \psi_0 - b_0^2 d_0 \ln(a_0 + c_0) + c_0 d_0 b_0 + c_0 \arcsin d_0 - 2d_0 \ln(a_0 + c_0)]. \qquad (42)$$

If we write an expression analogous to (28) for the effect of the ΔN_A section, we see that the following equations hold true for symmetric points of sections ΔN_A and ΔN_B:

$$(r_1)_A = (r_2)_B, \quad (180° - \beta)_A = (\alpha)_B,$$
$$(r_2)_A = (r_1)_B, \quad (180° - \beta)_B = (\alpha)_A.$$

Symbols A and B indicate, in this case, the central section in which a point is located. For this reason, in the expression

$$\Delta N_A = \int \Delta g \left[\frac{1}{2\pi\gamma} \left(\frac{1}{r_2} - \frac{1}{r_1} \right) - \frac{l}{2\pi\gamma} \left(\frac{\cos\alpha}{r_2^2} + \frac{\cos\beta}{r_1^2} \right) \right] d\Sigma$$

the multiplier in square brackets will be of a sign contrary to that of that multiplier in expression (28). While in section ΔN_A, the variation of the force of gravity is represented by the expression

$$\Delta g = \Delta g_A + \left(\frac{\partial g}{\partial x} \right)_A r_1 \cos\beta + \left(\frac{\partial g}{\partial y} \right)_A r_1 \sin\beta,$$

in the expression for the effect of section ΔN_A, paralleling expression (42), the sign of the first term (containing now Δg_A as a multiplier) will be inverted, while the sign of the second term will be preserved. The latter follows from the fact that in symmetric points of the ΔN_A and ΔN_B sections, anomaly variations resulting from the gradient $\frac{\partial g}{\partial x}$ will be equal in value and opposite in sign, when $\left(\frac{\partial g}{\partial x}\right)_A = \left(\frac{\partial g}{\partial x}\right)$

Thus

$$\frac{\Delta N_A}{l} = -\frac{2\Delta g_A}{\pi\gamma} \left[-b_0 \operatorname{arc\,tg} \frac{b_0 c_0}{a_0 d_0} + a_0 \operatorname{arc\,tg} \frac{d_0}{c_0} \right] + \frac{1}{\pi\gamma} \left(\frac{\partial g}{\partial x} \right)_A [-c_0 d_0 (a_0 + b_0) + \alpha_0 (a_0 c_0^2 - b_0 d_0^2) + b_0 d_0^3 \pi - c_0 a_0^2 \psi_0 - b_0^2 d_0 \ln (a_0 + c_0) + c_0 d_0 b_0 + c_0 \arcsin d_0 - 2 d_0 \ln (a_0 + c_0)].$$

By combining the expressions for $\frac{\Delta N_A}{l}$ and $\frac{\Delta N_B}{l}$, we get:

$$\frac{\Delta N_A + \Delta N_B}{l} = (\Delta g_B - \Delta g_A) \frac{2}{\pi\gamma} \left[-b_0 \operatorname{arc\,tg} \frac{b_0 c_0}{a_0 d_0} + a_0 \operatorname{arc\,tg} \frac{d_0}{c_0} \right] + \left[\left(\frac{\partial g}{\partial x} \right)_A + \left(\frac{\partial g}{\partial x} \right)_B \right] \frac{1}{\pi\gamma} [-c_0 d_0 (a_0 + b_0) + \alpha_0 (c_0^2 a_0 - b_0 d_0^2) + b_0 d_0^2 \pi - c_0 a_0^2 \psi^2 - b_0^2 d_0 \ln (a_0 + c_0) + c_0 d_0 b_0 + c_0 \arcsin d_0 - 2 d_0 \ln (a_0 + c_0)]. \quad (43)$$

(43)

Values may be assigned to a_o and b_o arbitrarily in (43), limiting their selection only by the consideration that variations of Δ_g within the area they define may be considered linear. Therefore, they must be chosen so that the multiplier for $(\Delta g_B - \Delta g_A)$ becomes equal to zero. This condition may be written as:

$$K_1 = -b_0 \operatorname{arc\,tg} \frac{b_0 c_0}{a_0 d_0} + a_0 \operatorname{arc\,tg} \frac{d_0}{c_0} = 0. \quad (I)$$

(I)

Molodenskiy adopted the following set of values, which satisfy approximately the conditions of the equation:

To these values correspond:

$$d_0 = 0{,}4274,$$
$$c_0 = 0{,}5774.$$

Since the values selected for c_o and d_o only come close to satisfying condition I, we obtain:

$\frac{2K_1}{\pi\gamma^2} = +0{,}266 \cdot 10^{-9}$, when γ is expressed in milligals.

For $\Delta g_B - \Delta g_A$ = 600 milligals, which is unlikely even in a highly anomalic mountainous area, errors in the elevation of the quasi-geoid amount to only 2 cm for l = 100 km. Such an error may be disregarded. For a coefficient L for the anomaly gradients $\left[\left(\frac{\partial g}{\partial x}\right)_A + \left(\frac{\partial g}{\partial x}\right)_B\right]$, we obtain from formula (43):

L = - 0.000 000 066 9506, if γ is expressed in milligals;
L" = ρ" L" = - 0". 013810, where ρ = 206 265".

Molodenskiy writes formula (43) somewhat differently. Specifically, he omits the last three terms in the expression for L as a result of the minuteness of their total. Using Molodenskiy's formula, we obtain

L" =-0".0143 80.

Molodenskiy's paper [1] gives -0."0151 as the value for L", apparently as a result of an error in calculation . In the case chosen by Molodenskiy, when c_o = 0.5774 and d_o = 0.4274, each of the sections is enclosed in a quadrangle with sides measuring 0.2506 along the x axis and 0.4936 along the y axis. For AB = 100 km (l = 50 km), the side on the x axis attains 12.5 km, while that on the y axis equals 25 km. The fact that the extension of the section along the y axis is twice that on the other has no great significance, since the anomalies situated above and below B do not affect significantly the component of the deviation of gravity of B in the direction of x. Molodenskiy did not find it efficient to reduce the area of the section, since the L function changes so rapidly in the niehgborhood of the pole (cf. expression 17) that larger errors might creep in in numerical integration than on the assumption of the linear variation of Δg within the section.

We have adopted the following set of values, satisfying equation I:

$$c'_o = 0.3569$$
$$d'_o = 0.3104$$

To this set of values correspond:

$$a'_o = 1.0618$$
$$b'_o = 0.9506.$$

Since our values for c'_o and d'_o approximate the condition of equation I, the value of

$$\frac{2K'}{\pi\gamma} = -0{,}214\cdot10^{-9},$$

if γ is expressed in milligals.

The coefficient L" for the gradients in our case equals

$$L'' = -0''.006335,$$

each of the sections being enclosed within a rectangle whose sides equal 0.1112 along the x axis and 0.2215 along the y axis. For AB = 100 km (1 = 50 km), the side on the x axis attains 5.5 km, that on the y axis, 11 km.

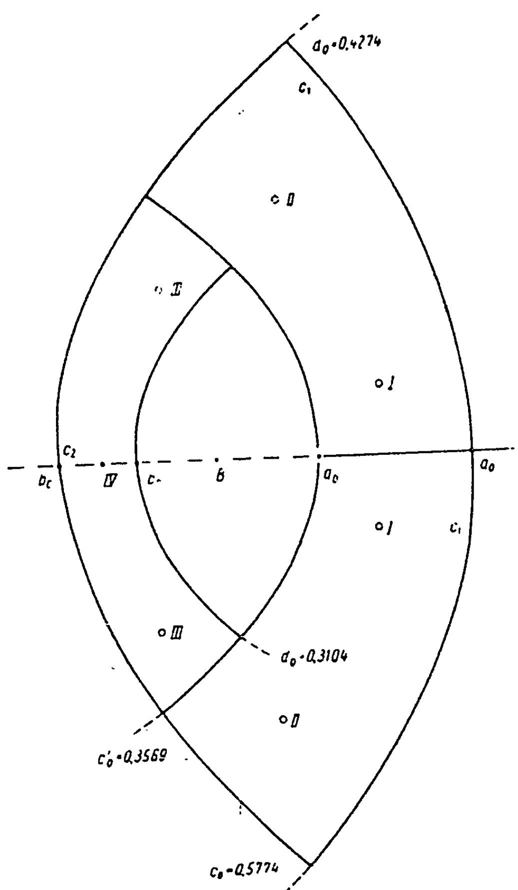

Fig. 1.

In Fig. 1, the curves $c_o = 0.5774$ and $d_o = 0.4274$ bound Molodenskiy's central section, while the curves c'_o = 0.3569 and d'_o = define our section.

The effect of the two sectors c_1 and c_2 which are equal in area may be calculated by means of the Gauss formula:

$$\frac{\Delta N(c_1)}{l} = \int\limits_{0}^{0,4274} \delta d \int\limits_{+0,3569}^{+0,5774} \Delta g\, M(a, b)\, \delta c = \frac{2}{\rho''} \sum_k A_k \Delta g_k,$$

The effect of the c_2 sector is determined by means of the following formula:

$$\frac{\Delta N(c_2)}{l} = \int\limits_{0,3104}^{0,4274} \delta d \int\limits_{-0,3569}^{+0,3569} \Delta g\, M(a, b)\, \delta c = \frac{2}{\rho''} \sum_k A_k \Delta g_k.$$

Values for A_k coefficients for varying numbers of ordinates are given below, with the x and y coordinates of corresponding points. X and y coordinates are expressed in units of l, while A_k coefficients are given in units of the fifth decimal figure. For the c_1 sector we obtained:

№ точек	1	2
c	c_1	c_1
d	d_1	d_2
A_k	− 1125	− 57
x	1,0992	1,0391
y	± 0,0422	± 0,1575
Δg	+49,6	+19,6

No. o[illegible] point[illegible]

№ точек	1	2	3	4
c	c_1	c_1	c_1	c_1
d	d_1	d_2	d_3	d_4
d_k	− 450	− 616	− 182	+ 27
x	1,1032	1,0927	1,0575	1,0127
y	± 0,0139	± 0,0659	± 0,1338	± 0,18[illegible]
Δg	+51,6	+46,4	+28,8	+ 6,3

No. o[illegible] point[illegible]

№ точек	1	2	3	4	5	6	7	8
c	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2
d	d_1	d_2	d_3	d_4	d_1	d_2	d_3	d_4
A_k	− 307	− 385	− 70	+ 36	−170	−248	− 95	0
x	1,0779	1,0676	1,0332	0,9894	1,1316	1,1208	1,0847	1,0387
y	±0,0120	± 0,0569	± 0,1155	± 0,1605	± 0,0158	± 0,0749	± 0,1520	± 0,2111
Δg	+38,9	+33,8	+16,6	− 5,3	+65,8	+60,4	+42,4	+19,4

Для участка c_2:

F[illegible]

№ точек	1	2
c	c_1, c_3	c_2
d	d_1	d_1
A_k	+315	+1952
x	0,9643	0,9295
y	± 0,1020	0
Δg	−17,8	− 35,3

№ точек	1	2	3
c	c_1, c_5	c_2, c_4	c_3
d	d_1	d_1	d_1
A_k	+ 89	+541	+1258
x	0,9769	0,9465	0,9295
y	± 0,1193	± 0,0709	0
Δg	−11,6	−26,8	−35,3

No of points	1	2	3	4
№ точек	1	2	3	4
c	c_1, c_7	c_2, c_6	c_3, c_5	c_4
d	d_1	d_1	d_1	d_1
A_k	+ 38	+176	+564	+918
x	0,9813	0,9615	0,9392	0,9295
y	± 0,1249	± 0,0976	± 0,0534	0
Δg	− 9,3	−19,3	−30,4	−85,3

No of points	1	2	3	4	5	6	7	8
№ точек	1	2	3	4	5	6	7	8
c	c_1, c_7	c_1, c_7	c_2, c_6	c_2, c_6	c_3, c_5	c_3, c_5	c_4	c_4
d	d_1	d_2	d_1	d_2	d_1	d_2	d_1	d_2
A_k	+ 12	+ 24	+ 78	+ 94	+308	+257	+550	+390
x	0,9948	0,9664	0,9746	0,9469	0,9520	0,9249	0,9422	0,9153
y	± 0,1135	± 0,1364	± 0,0887	± 0,1066	± 0,0485	± 0,0583	0	0
Δg	− 2,6	−16,8	−12,7	−26,5	−24,0	−37,6	−28,9	−42,3

The sign of A_k coefficients and values of x should be reversed for symmetrical points in sectors $c_1 + c_1$ and c_2 at point A. The values for the quantity Δg are explained below.

The effect of sectors $c_1 + c_1$ and c_2 equals the difference between the effects of areas bounded by the curves ($c_0 = 0.5774$, $d_0 = 0.4274$ and $c'_0 = 0.3569$, $d'_0 = 0.3104$). The difference between the effects of these areas, expressed in units of 1 for the case when Δg anomalies are constant in Molodenskiy's sections and $\Delta g_B = 1$ mgl, with $\Delta g_A = 0$, equals

$$H = \frac{2}{\pi\gamma}(K_1 - K'_1) = +10^{-9}(0{,}226 + 0{,}214) = 10^{-9} \cdot 0{,}480.$$

This difference must equal

$$\left(\frac{2}{\rho''}\sum_k A_k\right)_{2c_1} + \left(\frac{2}{\rho''}\sum_k A_k\right)_{c_2}.$$

For varying numbers of items in the expressions of $\frac{2}{\rho''}\sum_k A_k \Delta g_k$ for sectors $c_1 + c_1$ and c_2, we get:

Total number of points in area $c_1 + c_1 + c_2$	7	9	15	30
H	$10^{-9}\times$ +21,1	+14,9	+ 3,1	+0,78

When $\Delta g_B = 600$ mgl, $\Delta g_A = 0$ and $l = 100$ km, the corresponding error D in the elevation of the quasi-geoid (in meters) equals:

Total number of points in area $c_1 + c_1 + c_2$	7	9	15	30
D	+ 1,3	+ 0,9	+ 0,2	+0,05

For $\Delta g_B = 100$ mgl, $\Delta g_A = 0$ and $l = 50$ km, the error in the elevation of the quasi-geoid (in centimeters) equals:

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Total number of points in area $c_{1_D} + c_1 + c_2$	7	9	15	30
	+ 11	+ 7	+ 2	0

When the difference $\Delta g_B - \Delta g_A = 40$ mgl and $l = 50$ km (this case may be considered typical for flat areas), the error in the elevation of the quasi-geoid (in centimeters) equals:

Total Number of points in area $c_{1_D} + c_1 + c_2$	7	9	15	30
	+ 4	+ 3	+ 1	0

In this last case, the error is low enough when we use the Gauss formula with a small number of ordinates (7 points). The corresponding grid may be recommended for practical use. It should be remembered that the error in the result will be different for a different distribution of anomalies. The use of a grid with a greater number of points should provide an adequate control.

The calculations of the A_k coefficient for the $c_1 + c_1$ and c_2 sectors and of the coefficients for the anomalies of the force of gravity in the reduced central sections and Molodenskiy's central sections were checked in the following manner. Two astronomical points were taken on a flat Earth at a distance of $2\,l = 1000$ km from one another. The gravitational anomalies in all sectors equal zero. Only in the central section of Molodenskiy, containing astronomical point B, is the gravitational field accurately described by the gradient $\frac{\delta \Delta g}{\delta x} = 1 \frac{\text{mgl}}{\text{km}}$. At point B, $\Delta g_B = 0$.

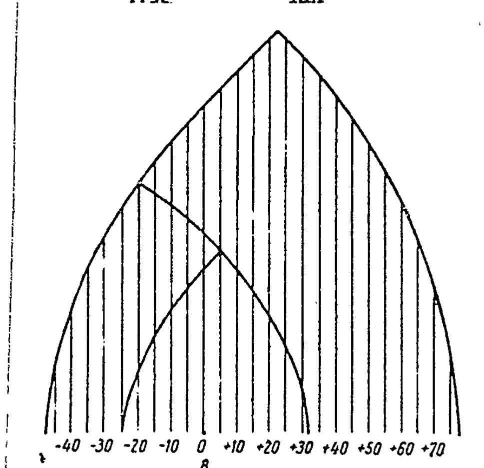

Fig. 2.

The effect of all sectors of Molodenskiy's section was determined in two ways: 1) by means of Yeremeyev's grid (the values of the coefficients of Yeremeyev's grid were used with a larger number of significant figures than appear in papers [3] and [2]), using the coefficients calculated for A_k and L". In using Yeremeyev's grid, the calculations were based on formula (28). A gravimetric map on a scale of 1: 1,000,000 (Fig. 2) was drawn for Molodenskiy's central section, containing point B. For purposes of plotting the map, coordinates were calculated for the points outlining areas $c_1 + c_1$, c_2, and the central sections. Yeremeyev's grid yielded the following results:

	$\frac{1}{2\pi\gamma}\int \Delta g \frac{1}{r_2} d\sum$	$\frac{\rho''}{2\pi\gamma}\int \Delta g \frac{\cos\alpha}{r_2^2} d\sum$
Molodenskiy's central section	+ 0,820 м	+ 7",23
Reduced central section	+ 0,077 .	+ 3",20
Sector $c_1 + c_1$	+ 1,122 .	+ 2",66
Sector c_2	− 0,379 .	+ 1",37

The effect of

$$\frac{1}{2\pi\gamma}\int \Delta g \frac{1}{r_1} d\sum \text{ и } \frac{1}{2\pi\gamma}\int \Delta g \frac{\cos\beta}{r_1^2} d\sum$$

may be easily found by approximation

$$\frac{1}{2\pi\gamma}\int \Delta g \frac{1}{r_1} d\sum = \frac{\Delta g_{cp.}\,\sigma}{2l\cdot 2\pi\gamma},$$

$$\frac{\rho''}{2\pi\gamma}\int \Delta g \frac{1}{r_1^2}\cos\beta\, d\sum = \frac{\Delta g_{cp.}\,\sigma\rho''}{2\pi\gamma\cdot 4l^2}.$$

where Δg_{cp} is the mean gravitational anomaly for the corresponding sector, and σ is the area of the sector.

In this manner, we obtained:

	$\frac{1}{2\pi\gamma}\int \Delta g \frac{1}{r_1} d\sum$	$\frac{\rho''}{2\pi\gamma}\int \Delta g \frac{\cos\beta}{r_1^2} d\sum$.
Molodenskiy's central section	+ 0,049 м	+ 0",01
Reduced central section	+ 0,003 .	0",00
Sector $c_1 + c_1$	+ 0,062 .	+ 0",01
Sector c_2	− 0,016 .	0",00

In units of the fifth decimal figure, we now obtain:

Molodenskiy's central section	− 3,36
Reduced central section	− 1,54
Sector $c_1 + c_1$	− 1,08
Sector c_2	− 0,74

Using the coefficients A_k and L", we get:

Molodenskiy's central section	− 3,35
Deviation, in percentage	0,3
Reduced central section	− 1,54
Deviation, percentage	0,0

Sector $c_1 + c_1$	– 1,14	– 1,14	– 1,12	
Number of points	4	8	16	
Deviation, percentage	6	6	4	
Sector c_2	– 0,80	– 0,76	– 0,74	– 0,74
Number of points	3	5	7	14
Deviation, percentage	8	3	0	0

In determining the effect of sectors $c_1 + c_1$ and c_2 from the Gauss formula, we used values for anomalies Δg in milligals given above. Deviations in the effect of sectors $c_1 + c_1$ and c_2 is to be explained by the inaccuracy of the Gauss formula, and decrease as the number of ordinates is increased. In using a grid with seven points, the highest deviation in the total effect of sectors $c_1 + c_1$ and c_2 will amount to 7% of that effect. In an area of flat relief, the effect of the entire Molodenskiy section can hardly exceed 0.50 m. In that case, the error in question will amount to 3.5 cm. We may resign ourselves to an error of such magnitude, considering that it is a limit value for an area of flat relief.

Thus, the formula for the effect of the central zone of Molodenskiy's elliptical grid may be written as follows:

$$\rho'' \frac{\Delta N_A + \Delta N_B}{2l} = -0'',00690 \left[\left(\frac{\partial g}{\partial x}\right)_A + \left(\frac{\partial g}{\partial x}\right)_B\right] l \quad \text{для отсеков Молоденского} \quad \text{II} \qquad \text{II}$$

or

$$\rho'' \frac{\Delta N_A + \Delta N_B}{2l} = -0'',00317 \left[\left(\frac{\partial g}{\partial x}\right)_A + \left(\frac{\partial g}{\partial x}\right)_B\right] l + \sum_k A_k \Delta g_k, \quad \text{III} \qquad \text{III}$$

where x and l in the left-hand portion are expressed in identical units. For practical purposes, another form for these formulas is preferable.

Finding the gradients for Molodenskiy's sectors in the following manner

$$\left(\frac{\partial g}{\partial x}\right)_B = \frac{\Delta g_{a_0} - \Delta g_{b_0}}{a_0 - b_0} = \frac{\Delta g_{a_0} - \Delta g_{b_0}}{0,250\,663}, \text{ так как since } a_0 = 1,154\,726,\ b_0 = 0,904\,063,$$

$$\left(\frac{\partial g}{\partial x}\right)_A = \frac{\Delta g_{b_0} - \Delta g_{a_0}}{b_0 - a_0} = \frac{\Delta g_{b_0} - \Delta g_{a_0}}{0,250\,663}, \text{ так как since } a_0 = -1,154\,726,\ b_0 = -0,904\,063.$$

and substituting into formula II, we get

$$\rho'' \frac{\Delta N_A + \Delta g_B}{2l} = +0''02755\,[(\Delta g_{a_0} - \Delta g_{b_0})_A + (\Delta g_{b_0} - \Delta g_{a_0})_B] \quad \text{IV} \qquad \text{IV}$$

After similar transformations, formula III will appear as fol[lows:]

$$\rho'' \frac{\Delta N_A + \Delta N_B}{2l} = +0'',02849\,[(\Delta g_{a'_0} - \Delta g_{b'_0})_A + (\Delta g_{b'_0} - \Delta g_{a'_0})_B] + \sum_k A_k \Delta g_k. \qquad \text{V}$$

It is desirable to use formulae IV and V jointly for purposes of verification when performing astronomical and gravimetric leveling on the basis of cluster surveys and a general gravimetric survey. In level areas, term $\sum_{k} A_k \Delta g_k$ may be composed of seven items. The corresponding grid is shown in Fig. 1.

In mountainous areas (when $k < 0.01$), a grid with a greater number of points should be used for comparison.

The effect of central sections in performing astronomic and gravimetric leveling on the basis of cluster surveys is gaged by means of gravimetric maps of the immediate vicinities of astronomical points. Such gravimetric maps are drawn on a scale of 1: 100,000. They must contain astronomical points, gravimetric points of the central and first zones of the cluster surveys, points for which Δg anomalies are found in determining the effect of sectors $c_1 + c_1$ and c_2 by means of the Gauss formula, and the points:

$$x = a_0 \qquad x = a'_0$$
$$x = -a_0 \qquad x = -a'_0$$
$$x = b_0 \qquad x = b'_0$$
$$x = -b_0 \qquad x = -b'_0.$$

The schedule of calculations given below (Table 1) may be recommended.

In the example given, the divergence of the results obtained for Molodenskiy's sections and for ours is negligible. However, in some cases the divergence has been known to attain 16 cm. The reason for such a great divergence was the incorrect determination of gradients $\frac{\partial g}{\partial x}$ in calculating the effect of Molodenskiy's sections: use was made of anomaly gradients at astronomical points, instead of mean gradients in the central sections between points $x = a_0$, $x = b_0$ and $x = -a_0$, $x = -b_0$, which should have been used.

Divergences in meters have been found for the effects of central sections in a single polygon of astronomic and gravimetric leveling (level area)(Table 2).

The area Σ, within which gravitational anomalies are taken into account in astro-gravimetric leveling, is divided by Molodenskiy into six sectors $S_1 - S_6$, in addition to the central sections defined by him. The effect of these sectors is determined from the Gauss numerical integration formula. Molodenskiy provides formulas [1] for the calculation of the A_k coefficients in various sectors. The elliptical grid of 1937 contains a large number of points in this portion of area Σ: they number 70 around each astronomical point. This must be considered adequate for both level and mountainous areas.

Molodenskiy has Σ bounded by an ellipse, whose small

Table 1

Points	Astron. pt. A		Astron. pt. B		$\lvert\Delta g\rvert$	A_k	$2l = 100{,}0$ км
	−	−	+	+			
1	+10	+14	+31	+29	+36	−1125	$\rho'' \frac{\Delta N_A + \Delta N_B}{2l} = -0'',48262$
2	+ 6	+28	+26	+28	+20	− 57	
1	+10	+25	+24	+28	+17	+ 315	$\Delta N_A + \Delta N_B = -0{,}23$ м
2	+18		+25		+ 7	+1952	
a'_0	+13		+29				
b'_0	+18		+25				
$\Delta g_{b'_0} - \Delta g_{a'_0}$	+ 5		− 4		− 9	+2849	
a_0	+ 9		+32				$\rho'' \frac{\Delta N_A + \Delta N_B}{2l} = -0'',52345$
b_0	+20		+24				
$\Delta g_{b_0} - \Delta g_{a_0}$	+11		− 8		−19	+2755	$\Delta N_A + \Delta N_B = -0{,}25$ м

Table 2

No. of links	For gradients of astron. pts.	For mean gradients	For mean gradients in reduced section and Gauss formula	l км
1	− 0,04	− 0,04	− 0,07	33
2	0,00	− 0,02	− 0,03	32
3	− 0,02	− 0,01	− 0,01	10
4	− 0,10	− 0,15	− 0,14	41
5	− 0,02	+ 0,06	+ 0,05	40
6	+ 0,01	0,00	0,00	12
7	− 0,37	− 0,22	− 0,21	44
8	0,00	− 0,02	− 0,06	44
9	0,00	0,00	0,00	7
10	+ 0,14	+ 0,13	+ 0,10	44
11	−	+ 0,03	+ 0,04	32

semi-axis c equals 2.8284 l, and whose long semi-axis a = 3 l. In his paper [2], Molodenskiy recommends lengthening the radius of area Σ to 4 l. Therefore, we will add four more sectors to Molodenskiy's area Σ, enclosed by an ellipse whose long semi-axis a = 4 l (Fig. 3).

The effect of sector S_7 on the elevation of the quasi-geoid $N_B - N_A$ may be written in the following form:

$$\frac{\Delta N(S_7)}{l} = \int_0^1 \delta d \int_{2{,}8284}^{3{,}8730} \Delta g\, M(a, b) \delta c.$$

In integrating for c, let us take two ordinates (n = 2), and for d, five ordinates (m = 5). The expression for $\Delta N(S_7)$ may then be transformed as follows:

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$$\frac{\Delta N(S_7)}{l} = 1{,}0446 \sum_{n=1}^{2} \sum_{m=1}^{5} A_n B_m \Delta g_{n,m} M(a_n, b_m) = \rho'' \sum_k A_k \Delta g_k,$$

The value of the subintegral function must be determined for

$$c_n = 2{,}8284 + 1{,}0446\, x_n$$
$$d_m = y_m.$$

We have repeated Molodenskiy's calculations and have been convinced of the accuracy of most of the coefficients. The small inaccuracies that do occur will be corrected in a table to be appended to the instruction manual on astro-gravimetric leveling now in preparation. In that table, sectors S_3 and S_7 will be combined.

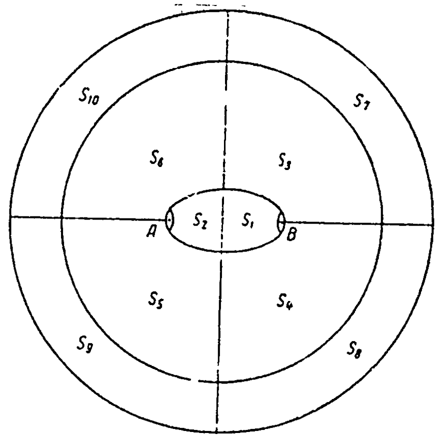

Fig. 3.

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THE SOLUTION OF THE INTEGRAL EQUATION DESCRIBING THE FORM OF THE EARTH

Works of the Central Scientific Research Institute of Geodesy, Aerial Surveying and Cartography, No 121, pp 41 -42

M. I. Yurkina

As shown by M. S. Molodenskiy [1], [2], the disturbance potential T may be determined with the aid of the auxiliary φ function, whose physical meaning denotes the density of a simple layer distributed over the surface S of an "Earth of the first approximation". The area of S is obtained by plotting from the reference surface only normal elevations H. In this connection,

$$T = \int_S \frac{\varphi}{r} dS, \qquad (1) \qquad (1)$$

where r is the distance from element dS to the point concerned. To determine φ we need to solve the integral equation:

$$2\pi\varphi \cos\alpha = (g-\gamma) + \frac{3}{2\rho_0}\int_S \frac{\varphi}{r} dS + \frac{1}{2\rho_0}\int_S \frac{\rho^2-\rho_0^2}{r^3}\varphi\, dS, \qquad (2) \qquad (2)$$

in which $g - \gamma$ is the gravitational anomaly measured on the physical surface of the Earth; α is the angle formed by radius vector ρ_0, drawn from the center of the surface of reference to the point where φ is to be determined; ρ is the radius vector of the field point of surface S. In his paper [2], M. S. Molodenskiy proposes to solve equation (2) by the isolation, in the gravitational field of the Earth, of a basic portion corresponding to the usual approximate solution. M. S. Molodenskiy proposes to describe this portion of the gravitational field by means of $g - \gamma$ anomalies, referred to the sphere of reference. The corresponding components of the elevation ζ of the cuasi-geoid and the deviation of the vertical at a point on the physical surface may be determined by means of the generalized Stokes formula and the generalized Waning-Meines formula. This was the procedure in calculations following Molodenskiy's formulae on a model [3], and in test calculations following the same formulae in the Crimea [4]. These researches have shown that in mountain areas (when the points under study are at significant altitudes), values for elevations of the quasi-geoid and deviations of the vertical are more nearly true if obtained from Stokes and Wening-Meines formulae than by means of generalized formulae. For this reason, it is advisable to define differently the basic portion of the Earth's gravitational field in some instances.

Let us draw a sphere Σ passing through the point under study, with a radius equal to $R + H_0$ and a center coincident with that of the reference sphere R. Let us then assume

a simple layer on sphere $R + H_0$, and have its density σ in the vicinity Σ_0 of the point under study in projections of elements of surface S equal $\frac{g - \gamma}{2\pi f}$.

The potential $\overline{T}$ of this layer at the point under study will eq[illegible]

$$\overline{T} = f\int_{\Sigma_0} \frac{\sigma}{r_1}\,d\Sigma = \frac{1}{2\pi}\int_{\Sigma_0} \frac{g-\gamma}{r_1}\,d\Sigma. \quad (3)$$

Allowing for a relative error of $\frac{H_0}{R}$ in the result, we may assume in our calculations that the radius of the sphere Σ equals R. The effects of $\overline{\xi}$ and $\overline{\eta}$ of the layer on the deviation of the vertical at the point under study are determined in

$$\overline{\xi} = -\frac{1}{2\pi\gamma}\int_{\Sigma_0} (g-\gamma)\frac{1}{r_1^2}\cos A\,d\Sigma,$$

$$\overline{\eta} = -\frac{1}{2\pi\gamma}\int_{\Sigma_0} (g-\gamma)\frac{1}{r_1^2}\sin A\,d\Sigma.$$

The values fo[illegible] mined from anomalies of the area around the point concerned by means of the Stokes and Wening-Meines formulae. Thus, the gravitational field of a simple layer will approach the gravitational field of the anomalies of the area involved. For this reason, it is convenient to consider the gravitational field of this layer as a first approximation of the Earth's gravitational field. The gravitational anomalies corresponding to the material layer of Σ_0 at points of the physical surface

$$\overline{\Delta g} = -\left(\frac{\partial \overline{T}}{\partial \rho} + \frac{2\overline{T}}{\rho}\right)_s.$$

When $H - H_0$, we have

$$\overline{\Delta g} = g - \gamma - \frac{2\overline{T}}{\rho},$$

since, in that case,

$$\frac{\partial \overline{T}}{\partial \rho} = -2\pi f\sigma = -(g-\gamma).$$

In adding to the [illegible] simple layer with a density of $-\sigma = -\frac{g}{2\pi f}$, allowance must be made for the displacement of the center of the Earth's mass. Residual anomalies δg may be determined by Molodenskiy's method [2]. The correction δ in the deviation of the vertical for the inclination of the Earth's physical surface in our case will be small. Its value will be:

$$\delta = \sim -\frac{2\pi}{\gamma}\cdot\frac{2\overline{T}}{\rho}\cos(n, \lambda),$$

where λ is a direction parallel to the reference surface in which the deviation of the vertical is determined.

In the determination of an approximation of the first order for the gravitational field of the Earth by Molodenskiy's method, the value of δ for Yeremeyev's model [3] attained 4".08. In our case, δ amounts to 0".10 for the same point.

a simple layer on sphere $R + H_0$, and have its density σ in the vicinity Σ_0 of the point under study in projections of elements of surface S equal $\frac{g - \gamma}{2\pi f}$.

The potential $\overline{T}$ of this layer at the point under study will equ

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Allowing for a relative error of $\frac{H_0}{R}$ in the result, we may assume in our calculations that the radius of the sphere Σ equals R. The effects of ξ and η of the layer on the deviation of the vertical at the point under study are determined in

$$\overline{\xi} = -\frac{1}{2\pi\gamma} \int_{\Sigma_0} (g - \gamma) \frac{1}{r_1^2} \cos A \, d\Sigma,$$

$$\overline{\eta} = -\frac{1}{2\pi\gamma} \int_{\Sigma_0} (g - \gamma) \frac{1}{r_1^2} \sin A \, d\Sigma.$$

The values for [illegible] ...mined from anomalies of the area around the point concerned by means of the Stokes and Wening-Meines formulae. Thus, the gravitational field of a simple layer will approach the gravitational field of the anomalies of the area involved. For this reason, it is convenient to consider the gravitational field of this layer as a first approximation of the Earth's gravitational field. The gravitational anomalies corresponding to the material layer of Σ_0 at points of the physical surface

$$\overline{\Delta g} = -\left(\frac{\partial \overline{T}}{\partial \rho} + \frac{2\overline{T}}{\rho}\right)_s.$$

When $H - H_0$, we have

$$\overline{\Delta g} = g - \gamma - \frac{2\overline{T}}{\rho},$$

since, in that case,

$$\frac{\partial \overline{T}}{\partial \rho} = -2\pi f \sigma = -(g - \gamma).$$

In adding to the ...simple layer with a density of $-\sigma = -\frac{g - \gamma}{2\pi f}$, allowance must be made for the displacement of the center of the Earth's mass. Residual anomalies δg may be determined by Molodenskiy's method [2]. The correction δ in the deviation of the vertical for the inclination of the Earth's physical surface in our case will be small. Its value will be:

$$\delta = \sim -\frac{2\pi}{\gamma} \cdot \frac{2\overline{T}}{\rho} \cos (n, \lambda),$$

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a simple layer on sphere $R + H_0$, and have its density σ in the vicinity Σ_0 of the point under study in projections of elements of surface S equal $\frac{g - \gamma}{2\pi f}$.

The potential $\overline{T}$ of this layer at the point under study will equal

$$\overline{T} = f \int_{\Sigma_0} \frac{\sigma}{r_1} d\Sigma = \frac{1}{2\pi} \int_{\Sigma_0} \frac{g - \gamma}{r_1} d\Sigma. \qquad (3)$$

Allowing for a relative error of $\frac{H_0}{R}$ in the result, we may assume in our calculations that the radius of the sphere Σ equals R. The effects of $\overline{\xi}$ and $\overline{\eta}$ of the layer on the deviation of the vertical at the point under study are determined in

$$\overline{\xi} = -\frac{1}{2\pi\gamma} \int_{\Sigma_0} (g - \gamma) \frac{1}{r_1^2} \cos A \, d\Sigma,$$

[illegible] $d\Sigma.$

The values for $\overline{T}$, $\overline{\xi}$ and $\overline{\eta}$ are close to those determined from anomalies of the area around the point concerned by means of the Stokes and Wening-Meines formulae. Thus, the gravitational field of a simple layer will approach the gravitational field of the anomalies of the area involved. For this reason, it is convenient to consider the gravitational field of this layer as a first approximation of the Earth's gravitational field. The gravitational anomalies corresponding to the material layer of Σ_0 at points of the physical surface

$$\overline{\Delta g} = -\left(\frac{\partial \overline{T}}{\partial \rho} + \frac{2\overline{T}}{\rho}\right)_s.$$

When $H - H_0$, we have

$$\overline{\Delta g} = g - \gamma - \frac{2\overline{T}}{\rho},$$

since, in that case, [illegible]

In adding to the real Earth a simple layer with a density of $-\sigma = -\frac{g - \gamma}{2\pi f}$, allowance must be made for the displacement of the center of the Earth's mass. Residual anomalies δg may be determined by Molodenskiy's method [2]. The correction δ in the deviation of the vertical for the inclination of the Earth's physical surface in our case will be small. Its value will be

[illegible]),

where λ is a direction parallel to the reference surface in which the deviation of the vertical is determined.

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DETERMINATIONS OF A GRID FOR THE COMPUTATION OF ELEVATIONS OF THE QUASI-GEOID AND DEVIATIONS OF THE VERTICAL FROM THE FORMULAS OF STOKES AND WENING-MEINES

V. F. Yeremeyev

Works of the Central Scientific Research Institute of Geodesy, Aerial Surveying and Cartography, No 121, pp 43 - 75

A grid computed at the CSRIGASC in 1941 is currently in use for numerical integration in the calculation of deviations ξ and η of the vertical and elevations ζ of the quasi-geoid. The grid is designed for the use of the Stokes and Wening-Meines formulae, and is described in a paper by the author in Collected papers on Geodesy, No. 8, 1945. The grid is designed for determinations of ξ and η with an accuracy to 0".1, and has the following properties:

1. For taking account of the effect of anomaly fields outside the central area containing the point concerned, the area is divided into sectors (spherical trapezia). The mean value for the gravitational anomaly must be known for the area of each sector.

2. Within each of four groups of annular zones with approximate radii of 5 to 100 km, 100 to 300 km, 300 to 1000 km and 1000 to 2000 km, the effect of all rings on the deviation of the vertical is equal. The coefficients describing the effect of rings of various groups on the deviation of the vertical decrease as the group is further removed from the center of the grid. The trapezia within each ring differ in area and, therefore, affect ξ and η unequally, namely, in a manner proportional to the cosine or sine of the azimuth.

3. The central zone within the 5 km radius is accounted for separately, by means of a grid especially designed on the basis of numerical integration formulae of Lagrange and Gauss (for points). The grid for computing the effect of the central zone is designed in three variants: a) with one radius, for an even anomaly field with a small gradient; b) with three radii, for a more complex anomaly field; and c) with five radii, for unusually complex anomaly fields.

The grid described has been widely used, its method of calculation has not yet been published. The present article presents these calculations, which are of methodological interest, and demonstrate the sufficient accuracy of the grid.

I. Calculation of Grid for the Computation of Deviations of the Vertical

As we know

$$\zeta = \frac{R}{2\pi\gamma_0}\int_0^{\pi}\int_0^{2\pi}(g-\gamma)\,F(\psi)\,d\psi\,d\alpha, \qquad (1)$$

$$\xi = \frac{1}{2\pi\gamma_0}\int_0^{\pi}\int_0^{2\pi}(g-\gamma)\,Q(\psi)\cos\alpha\,d\psi\,d\alpha, \qquad (2)$$

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A grid computed at the CSRIGASC in 1941 is currently in use for numerical integration in the calculation of deviations ξ and η of the vertical and elevations ζ of the quasi-geoid. The grid is designed for the use of the Stokes and Wening-Meines formulae, and is described in a paper by the author in Collected papers on Geodesy, No. 8, 1945. The grid is designed for determinations of ξ and η with an accuracy to 0".1, and has the following properties:

1. For taking account of the effect of anomaly fields outside the central area containing the point concerned, the area is divided into sectors (spherical trapezia). The mean value for the gravitational anomaly must be known for the area of each sector.

2. Within each of four groups of annular zones with approximate radii of 5 to 100 km, 100 to 300 km, 300 to 1000 km and 1000 to 2000 km, the effect of all rings on the deviation of the vertical is equal. The coefficients describing the effect of rings of various groups on the deviation of the vertical decrease as the group is further removed from the center of the grid. The trapezia within each ring differ in area and, therefore, affect ξ and η unequally, namely, in a manner proportional to the cosine or sine of the azimuth.

3. The central zone within the 5 km radius is accounted for separately, by means of a grid especially designed on the basis of numerical integration formulae of Lagrange and Gauss (for points). The grid for computing the effect of the central zone is designed in three variants: a) with one radius, for an even anomaly field with a small gradient; b) with three radii, for a more complex anomaly field; and c) with five radii, for unusually complex anomaly fields.

The grid described has been widely used, its method of calculation has not yet been published. The present article presents these calculations, which are of methodological interest, and demonstrate the sufficient accuracy of the grid.

I. Calculation of Grid for the Computation of Deviations of of the Vertical

As we know,

(1)

$$\xi = \frac{1}{2\pi\gamma_0} \int_0 \int_0 (g-\gamma)\, Q(\psi) \cos\alpha \, d\psi \, d\alpha, \qquad (2)$$

(2)

$$\eta = \frac{1}{2\pi\gamma_0} \int_0^{\pi} \int_0^{2\pi} (g - \gamma)\, Q(\psi) \sin\alpha \, d\psi \, d\alpha, \qquad (3)$$

3)

where

$$F(\psi) = \frac{1}{2} \sin\psi \left[\csc\frac{\psi}{2} - 6\sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right],$$

$$Q(\psi) = -\frac{1}{2}\cos^2\frac{\psi}{2}\left[\csc\frac{\psi}{2} + 12\sin\frac{\psi}{2} - 32\sin^2\frac{\psi}{2} + \frac{3}{1+\sin\frac{\psi}{2}} - 12\sin^2\frac{\psi}{2}\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right].$$

R is the mean radius of the Earth, γ_0 is the mean value of normal gravity on Earth, g is the measured force of gravity, γ is the normal value corresponding to the measured g (calculated from the normal elevation and latitude of the point), ψ is the angular distance between the point under study and the field point, and α is the aximuth of the field point.

In solving practical problems, one is confined to determining the effect of a limited area on ξ, η and ζ. For $\psi < \frac{1000}{R}$, it may be posited that

$$Q = -\left(\frac{A}{r} + B + Cr\right),$$

where

$$r = 2R\sin\frac{\psi}{2}, \quad A = 1339'',6, \quad B = 0'',315, \quad C = 0'',000066,$$
$$R = 6371 \text{ км.}$$

Let us single out a central zone of radius $r_0 = 5$ km, whose effect on the deviation of the vertical we will calculate separately.

Then

$$\xi = \frac{1}{2\pi R} \int_{r_0}^{R_0} \int_0^{2\pi} \Delta g \, Q_1(r) \cos\alpha \, dr \, d\alpha \qquad (4)$$

(4)

or, replacing the double integral by the final sums,

$$\xi = \frac{1}{2\pi R} \sum_i \sum_k \Delta g_{ik} \, Q_1(r) \cos\alpha_i \, \Delta r \, \Delta\alpha. \qquad (5)$$

(5)

Let us now resolve the interval of integration from r_0 to R_0 into three annular areas, roughly from 5 to 100 km, from 100 to 300 km and from 300 to 1000 km.

We may then split up each annular area into concentric

$$\int^{\pi}\int^{2\pi} (g-\gamma)\, Q(\psi) \sin\alpha \, d\psi \, d\alpha, \qquad (3)$$ (3)

where

$$\ldots 5 \sin\frac{\psi}{2} + 1 - 5\cos\psi - \ldots \frac{\psi}{2} + \sin^2\frac{\psi}{2}\Big)\Big],$$

$$\ldots n\frac{\psi}{2} - 32\sin^{3}\frac{\psi}{2} + \frac{3}{1+\sin\frac{\psi}{2}} - \ldots \frac{\psi}{2} + \sin^{3}\frac{\psi}{2}\Big)\Big];$$

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Let us now resolve the interval of integration from r_0 to R_0 into three annular areas, roughly from 5 to 100 km, from 100 to 300 km and from 300 to 1000 km.

We may then split up each annular area into concentric zone

$$\int_{r_0}^{r_1} Q_1\,dr = \int_{r_1}^{r_2} Q_1\,dr = \ldots = \text{const.} = P$$

or,

$$\overline{Q_1}\,\Delta r = P = \text{const},$$

where $\overline{Q_1}$ is the mean for each zone. It is clear that for this ·urpose, we need to compute the radii of the zones from the formula

$$\ln r_k = \ln r_0 + \frac{B}{A} r_0 + \frac{C}{2A} r_0^2 + \frac{P}{A} k - \frac{B}{A} r_k - \frac{C}{2A} r_k^2 \qquad (6)$$

or, in common logarithms,

$$\lg r_k = \lg r_0 + \frac{MB}{A} r_0 + \frac{MC}{2A} r_0^2 + \frac{MP}{A} k - \frac{MB}{A} r_k - \frac{MC}{2A} r_k^2. \qquad (6')$$

By dividing the area of integration from 5 to 100 km into 16 equal sections, and the remaining two annular areas into 24 equal sections, we obtain

$$\xi'' = \frac{P''_1}{16\,R} \sum_i \sum_k \Delta g_{ik} \cos \alpha_i + \frac{P''_2}{24\,R} \sum_i \sum_k \Delta g_{ik} \cos \alpha_i + \frac{P''_3}{24\,R} \sum_i \sum_k \Delta g_{ik} \cos \alpha_i. \qquad (7)$$

Let us take

$$\frac{P''_1}{16\,R} = -0'',005, \quad \frac{P''_2}{24\,R} = -0'',002, \quad \frac{P''_3}{24\,R} = -0'',0015$$

and calculate by means of sequential approximations the value of radii r_k from formula (6'), reduced to the working form

$$\lg r_k = 0{,}69948 + 0{,}000324\,kP - 0{,}000102\,r_k - 0{,}000000011\,r_k^2, \qquad (8)$$

where r_k is expressed in kilometers.

At the same time, we may calculate the radii of the grid zones for a flat anomaly field from the formula

$$\lg r_k = \lg r_0 + \frac{MP}{A} k, \qquad (9)$$

which is written in that form because, for a "flat Earth", we have

$$Q = \frac{A}{r}.$$

These values will be needed in subsequent checking operations.

The following integrals have been calculated for determining the effect of the trapezia for $\frac{1000}{R}$ km

$$\int_{\psi_1}^{\psi_2} S(\psi)\sin\psi\, d\psi = \left[-\cos\psi + \frac{7}{4}\cos^2\psi + 2\sin\frac{\psi}{2}\left(\frac{3}{2}\cos\psi + \frac{1}{2}\right) - \frac{3}{2}\sin^2\psi \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right]\Bigg|_{\psi_1}^{\psi_2},$$

$$\int_{\psi_1}^{\psi_2} \frac{d}{d\psi} S(\psi)\sin\psi\, d\psi = \left[\frac{7}{4}\psi + \cos\frac{\psi}{2} - \frac{13}{2}\sin\frac{\psi}{2}\cos\frac{\psi}{2} - 6\sin^2\frac{\psi}{2}\cos\frac{\psi}{2} + 13\sin^3\frac{\psi}{2}\cos\frac{\psi}{2} - 2\ln\operatorname{tg}\frac{\psi}{4} + \left(\frac{3}{2}\psi - 3\sin\frac{\psi}{2}\cos\frac{\psi}{2} + 6\sin^3\frac{\psi}{2}\cos\frac{\psi}{2}\right)\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right]\Bigg|_{\psi_1}^{\psi_2} - 3\int_{\psi_1}^{\psi_2}\frac{\psi}{2}\frac{\cos\frac{\psi}{2}}{\sin\frac{\psi}{2}}d\frac{\psi}{2} + 3\int_{\psi_1}^{\psi_2}\frac{\psi}{2}\frac{\sin\frac{\psi}{2}}{\cos\frac{\psi}{2}}d\frac{\psi}{2} - 3\int_{\psi_1}^{\psi_2}\frac{\psi}{2}\frac{1}{\cos\frac{\psi}{2}}d\frac{\psi}{2}.$$

We may represent the sum of the last three terms by the series

$$-3\int_{\psi_1}^{\psi_2}\frac{\psi}{2}\frac{\cos\frac{\psi}{2}}{\sin\frac{\psi}{2}}d\frac{\psi}{2} + 3\int_{\psi_1}^{\psi_2}\frac{\psi}{2}\frac{\sin\frac{\psi}{2}}{\cos\frac{\psi}{2}}d\frac{\psi}{2} - 3\int_{\psi_1}^{\psi_2}\frac{\psi}{2}\frac{1}{\cos\frac{\psi}{2}}d\frac{\psi}{2} = \left[-\frac{3}{2}\psi - \frac{3}{8}\psi^2 + \frac{1}{6}\psi^3 - \frac{3}{128}\psi^4 + \frac{1}{150}\psi^5 - \frac{5}{3072}\psi^6 + \frac{1}{2205}\psi^7 - \frac{61}{491520}\psi^8 \ldots\right]\Bigg|_{\psi_1}^{\psi_2}.$$

Table 1

Radii of zones in kilometers

No of zones	For flat field of anomalies	For spherical fd of anomal.	No of zones	For flat fd of an	For spher fd of an
I	5,0	5,0	XIV		
II	7,3	7,3	XV	389,9	357,8
III	10,7	10,7	XVI	462,7	418,1
IV	15,7	15,7	XVII	549,2	487,4
V	22,9	22,8	XVIII	657,7	566,6
VI	33,5	33,3	XIX	773,5	656,6
VII	49,0	48,5	XX	917,9	758,0
VIII	71,7	70,6	XXI	1089,4	872,0
IX	104,9	102,6	XXII	1292,9	1000,0
X	131,8	128,0	XXIII		1163,7
XI	165,7	159,6	XXIV		1345,5
XII	208,1	198,6	XXV		1545,6
XIII	261,5	246,7	XXVI		1763,9
	328,6	305,4			2000,0

For $\psi \leq \frac{2000\ km}{R}$, we may confine ourselves to taking into account the term containing ψ at the fifth power. This will ensure a correct value fo[illegible] ...cimal in the value for the integral $\int_{\psi_1}^{\psi_2} \frac{d}{d\psi} S(\psi) \sin\psi \, d\psi$

Values for this integral have been calculated for $\psi = \frac{1000 \text{ km} + x}{R}$, where x varies by intervals of 50 km from 0 to 1000 km. Interpolating on the basis of every third difference, the radii were determined for the five zones in one sector of each of which trapezia affect the deviation of the vertical equally.

Each of these zones is divided by the radii into 48 trapezia of equal area (in this case, the trapezia are close to being squares).

The radii obtained by this means for zones from 5 to 2000 km are given in Table 1. In an earlier paper* the author provides erroneous values for the radii.

The value of the ξ deviation of the vertical in latitude determined by means of the grid with the values listed in Table 1 for an anomaly field without a central zone of a radius r_0 = 5 km is expressed by the following formula:

$$\xi = -0'',005 \sum_{n=I}^{n=VIII} \sum_{k=1}^{16} \Delta g_{nk} \cos\alpha_k - 0'',002 \sum_{n=IX}^{n=XIII} \sum_{k=1}^{24} \Delta g_{nk} \cos\alpha_k -$$

$$- 0'',0015 \sum_{n=XIV}^{n=XXI} \sum_{k=1}^{24} \Delta g_{nk} \cos\alpha_k - 0'',000\,871 \sum_{n=XXII}^{n=XXVI} \sum_{k=1}^{48} \Delta g_{nk} \cos\alpha_k; \quad (I)$$

$$\text{for } k=16 \ \alpha_k=\frac{\pi k}{8}, \ \text{for } k=24 \ \alpha_k=\frac{\pi k}{12}, \ \text{for } k=48 \ \alpha_k=\frac{\pi k}{24}.$$

The deviation η of the vertical in the plane of the initial vertical is expressed by a similar formula, in which the cosines are replaced by sines. In formula I, Roman numerals designate zones, Δg_{nk} is the mean value of anomalies in a trapezium whose mean azimuth equals α_k. We will designate as k the ordinal number of the section. The values of $\cos\alpha_k$ (or $\sin\alpha_k$, if we are calculating η) repeat themselves periodically with identical or inverse signs, and for this reason the products $\Delta g_{nk}\cos\alpha_k$ (or $\Delta g_{nk}\sin\alpha_k$) may be combined into groups with equal values of the cosine (or sine).

The effect of the central zone on the deviation of the vertical may be written thus:

$$\xi = -\frac{x''}{2\pi\gamma}\int_0^{r_0}\int_0^{2\pi} \frac{\Delta g}{r} \cos\alpha \, dr \, d\alpha = \frac{-x''}{2\pi\gamma}\int_0^{r_0}\left[\int_0^{2\pi} \Delta g \cos\alpha \, d\alpha\right]\frac{dr}{r}. \quad (10)$$

Let us divide the interval from 0 to r into two parts,

* V. F. Yeremeyev. The Calculation of Corrections for the Deviation of Vertical Lines for the Astronomical Coordinates of Points Used as Data for Topographic Surveys on a Small Scale. Collected Papers of the Main Office of Geodesy and Cartography, No 8, 1945.

so that within the central area of radius ρ we may take:

$$\Delta g = \Delta g_0 + \left(\frac{\partial g}{\partial r}\right)_0 r, \qquad (11) \quad (11)$$

where Δg_0 is the value of the anomaly at the point under study ($r = 0$), while $\left(\frac{\partial g}{\partial r}\right)_0$ is the value of the derivative or gradient of the anomaly along a given radius in the interval from 0 to ρ taken as constant.

Let us $f(r) = \int_0^{2\pi} \Delta g \cos\alpha \, d\alpha$. Then

$$\xi'' = -\frac{\varkappa''}{2\pi\gamma}\int_0^{\rho} f(r)\frac{dr}{r} - \frac{\varkappa''}{2\pi\gamma}\int_{\rho}^{r_0} f(r)\frac{dr}{r} = J_1 + J_2 \qquad (12) \quad (12)$$

Substituting the value for Δg from (11) into $f(r)$ under the first integral of J_1, we obtain

$$f(r) = \int_0^{2\pi}\left(\frac{\partial g}{\partial r}\right)_0 r\cos\alpha \, d\alpha \text{ и } J_1 = -\frac{\varkappa''}{2\pi\gamma}\int_0^{2\pi}\Delta g_{\rho}\cos\alpha \, d\alpha = -\frac{\varkappa''}{2\pi\gamma} f(\rho), \qquad (13) \quad (13)$$

where Δg_{ρ} is the value of the anomaly at a distance ρ from the point.

In the particular case when the anomalies around the point under study are relatively even, it may be assumed that $\rho = 5$ km, and we may divide the circumference of the 5 km radius into 8 equal portions. Then

$$\xi'' = -\frac{\varkappa''}{2\pi\gamma} f(r_0) = -\frac{\varkappa''}{8\gamma}\sum_{k=1}^{8}\Delta g_{ok}\cos\alpha_k = -0'',02628\sum_{k=1}^{8}\Delta g_{ok}\cos\alpha_k \qquad (II) \quad (II)$$

where Δg_{ok} is the value of anomaly at the circumference of radius $r_0 = 5$ km at the k-th point, $\alpha_k = \frac{\pi k}{4}$.

Since anomalies vary in a linear fashion within the interval from 0 to ρ, formula (13) may be written

$$J_1 = -\frac{\varkappa''}{2\pi\gamma} f(\rho) = -\frac{\varkappa''}{2\pi\gamma}\frac{\rho}{r} f(r), \qquad (13') \quad (13')$$

allowing thereby the variation of the radius of the circumference along which the distribution of anomalies must be known.

Let us clarify the significance of the second integral of formula (12)

$$J_2 = -\frac{\varkappa''}{2\pi\gamma}\int_{\rho}^{r_0} f(r)\frac{dr}{r}.$$

From the two-ordinate Lagrange numerical integration formula

$$J_2 = -\frac{\varkappa''}{2\pi\gamma}\int_{\rho}^{r_0} f(r)\frac{dr}{r} \approx -\frac{\varkappa''}{2\pi\gamma}(r_0 - \rho)\left[A_2 F(r_2) + A_3 F(r_3)\right], \qquad (14)$$

wh $F(r)=\frac{f(r)}{r}, \quad A_2+A_3=1, \quad A_2x_2+A_3x_3=0, \quad x=\frac{2r-(r_0+\rho)}{R_0-\rho}$. (15) (15)

By ... up (15), we get

$$A_2\frac{2r_2-(r_0+\rho)}{r_0-\rho}+A_3\frac{2r_3-(r_0+\rho)}{r_0-\rho}=0$$

or

$$\frac{2(A_2r_2+A_3r_3)}{r_0-\rho}-(A_2+A_3)\frac{r_0+\rho}{r_0+\rho}=0;$$

whence we fin $A_2r_2+A_3r_3=\frac{r_0+\rho}{r}$, , and therefore

$$A_2=\frac{2r_3-(r_0+\rho)}{2(r_3-r_2)}, \quad A_3=-\frac{2r_2-(r_0+\rho)}{2(r_3-r_2)}.$$

Subs), we have:

$$J_2=-\frac{\varkappa''}{4\pi\gamma}\cdot\frac{r_0-\rho}{(r_3-r_2)}\cdot\frac{2r_3-(r_0-\rho)}{r_2}-\frac{\varkappa''}{4\pi\gamma}\cdot\frac{r_0-\rho}{(r_3-r_2)}\times$$
$$\times\frac{r_0+\rho-2r_2}{r_3}f(r_3). \quad (16)$$

(16)

J_1 and J_2 of formula (12), we may find the effect an anomaly field with a radius r_0 = 5 km on the deviation of vertical lines, if the field is divided into fourzones by three circumferences described by radii r_1, r_2 and r_3:

$$\xi''=-\frac{\varkappa''}{2\pi\gamma}\frac{\rho}{r_1}f(r_1)-\frac{\varkappa''}{4\pi\gamma}\cdot\frac{r_0-\rho}{(r_3-r_2)}\cdot\frac{2r_3-(r_0+\rho)}{r_2}f(r_2)-$$
$$-\frac{\varkappa''}{4\pi\gamma}\cdot\frac{r_0-\rho}{(r_3-r_2)}\cdot\frac{r_0+\rho-2r_2}{r_3}f(r_3). \quad (17)$$

(17)

In the particular case with 8 sections, the formula takes the form

$$\xi''_8=-\frac{\varkappa''}{8\gamma}\frac{\rho}{r_1}\sum_{k=1}^{8}\Delta g(r_1)\cos\alpha_k-\frac{\varkappa''}{16\gamma}\cdot\frac{r_0-\rho}{r_3-r_2}\times$$
$$\times\frac{2r_3-(r_0+\rho)}{r_2}\sum_{k=1}^{8}\Delta g(r_2)\cos\alpha_k-\frac{\varkappa''}{16\gamma}\cdot\frac{r_0-\rho}{r_3-r_2}\times$$
$$\times\frac{r_0+\rho-2r_2}{r_3}\sum_{k=1}^{8}\Delta g(r_3)\cos\alpha_k. \quad (18)$$

(18)

For convenience in computation, we may take the following numerical values for the coefficients:

$$\frac{\varkappa''\rho}{8\gamma r_1}=0'',03; \quad \frac{\varkappa''}{16\gamma}\cdot\frac{r_0-\rho}{r_3-r_2}\cdot\frac{2r_3-(r_0+\rho)}{r_2}=0'',03;$$
$$\frac{\varkappa''}{16\gamma}\cdot\frac{r_0-\rho}{r_3-r_2}\cdot\frac{r_0+\rho-2r_2}{r_3}=0'',003. \quad (19)$$

(19)

By finding suitable values for ρ and solving the system of equations in (19), we find radii r_1, r_2 and r_3. A convenient subdivision of the 5 km area into zones results from ρ = 1.2 km. For that case, we find from (19): r_1 = 1.051 km, r_2 = 2.838 km, r_3 = 4.516 km. In final form, the formula for determining the effect of an anomaly field of average complexity and a radius of 5 km on the deviation of the vertical will be written as follows:

$$\xi''_8 = -0'',03 f(r_1) - 0'',03 f(r_2) - 0'',003 f(r_3). \quad \text{(III)}$$

(III)

For a complex anomaly field in a central area of a radius of 5 km, the grid is calculated by means of the Gauss formula. As in the preceding case, we segregate from the area of 5 km radius a small central portion of radius ρ , and break down the integral (10) into two (12). The solution of the first integral

$$J_1 = -\frac{\varkappa''}{2\pi\gamma}\frac{\rho}{r} f(r)$$

has already been found in calculating the preceding grid, and r may assume any value close to ρ , since the relation $\frac{f(r)}{r}$ near the center of the area is, for practical purposes, constant.

To calculate the second integral J_2, i.e. to allow for the effect the remaining annular area between radii ρ and $R_0 = 5$ km, we may use the numerical integration method following the Gauss formula with five ordinates

$$J_2 = -\frac{\varkappa''}{2\pi\gamma}\int_{\rho}^{r_0} f(r)\frac{dr}{r} = -\frac{\varkappa''}{2\pi\gamma}(r_0 - \rho)[A_1F(r_1) + A_2F(r_2) + A_3F(r_3) + A_4F(r_4) + A_5F(r_5)], \quad (20)$$

(20)

where

$$F(r_n) = \frac{1}{r_n} f(r_n), \quad r_n = \rho + (r_0 - \rho)x_n, \quad (21)$$

(21)

x_n is the value of the abcissa at the points of division of the integral following Gauss, and A_n are the coefficients in the Gauss formula. The numerical values for x_n and A_n for $n = 1, 2 \ldots 5$ are as follows

$$x_1 = 0,04691 \qquad A_1 = A_5 = 0,11846$$
$$x_2 = 0,23077 \qquad A_2 = A_4 = 0,23931$$
$$x_3 = 0.50000$$
$$x_4 = 0,76923 \qquad A_3 = 0,28444$$
$$x_5 = 0,95309$$

If the area is divided into 12 sections, i.e. if we posit $\Delta\alpha = \frac{\pi}{6}$ and take a value for ρ in J_1 so as to have it coincide with that of r_1 in J_2, the sum of J_1 and J_2 for our 5-km area may be written thus:

$$\Delta\xi'' = J_1 + J_2 = -a_1 f(r_1) - a_2 f(r_2) - a_3 f(r_3) - a_4 f(r_4) - a_5 f(r_5). \quad (22)$$

Here

$$f(r_n) = \sum_{k=1}^{k=12} \Delta g(r_n)\cos\alpha_k \; (k = 1, 2, 3, 4 \ldots 12, \quad n = 1, 2, 3, 4, 5),$$

$$a_1 = \frac{\varkappa''}{12\gamma}\left(\frac{\rho}{r_1} + \frac{r_0-\rho}{r_1}A_1\right), \qquad a_4 = \frac{\varkappa''}{12\gamma}\,\frac{r_0-\rho}{r_4}A_4$$

$$a_2 = \frac{\varkappa''}{12\gamma}\left(\frac{r_0-\rho}{r_2}A_2\right),$$

$$a_3 = \frac{\varkappa''}{12\gamma}\cdot\frac{r_0-\rho}{r_3}A_3 \qquad a_5 = \frac{\varkappa''}{12\gamma}\,\frac{r_0-\rho}{r_5}A_5.$$

Taking $\rho = 0.2$ km and $r_0 = 5$ km, it is possible to find the numerical values for a_1, a_2, a_3, a_4, a_5 and r_1, r_2, r_3, r_4, r_5:

$$\begin{aligned} a_1 &= 0'',03167, & r_1 &= 0{,}425 \text{ км} \\ a_2 &= 0\ ,01539, & r_2 &= 1{,}308 \text{ „} \\ a_3 &= 0\ ,00920, & r_3 &= 2{,}600 \text{ „} \\ a_4 &= 0\ ,00517, & r_4 &= 3{,}892 \text{ „} \\ a_5 &= 0\ ,00209, & r_5 &= 4{,}775 \text{ „} \end{aligned}$$

In final form, the formula for finding the effect of an anomaly field of a radius of 5 km with the numerical coefficients given on the deviation of the vertical will appear as follows:

$$\Delta\xi'' = -0'',03167\,f(r_1) - 0'',01539\,f(r_2) - 0'',00920\,f(r_3) - 0'',00517\,f(r_4) - 0'',00209\,f(r_5). \qquad \text{(IV)}$$

II. Verification of the Calculation of Grids for ξ and η by Means of Artificial Models

Grid for Flat Anomaly Field

To verify the correctness of the calculation of the grid and all the computations connected therewith, it is desirable to test them by means of an anomaly field whose effect on the deviation of vertical lines is known exactly. By comparing theoretical values for the deviation of the vertical with those computed by means of the grid, it is possible to test the correctness of the computation and the accuracy of the grid, if the anomaly field chosen is sufficiently distinctive, i.e. if the gradient and magnitude of the anomalies of the field chosen are sufficiently pronounced and the averaging of anomalies by trapezia involves certain difficulties.

A suitable anomaly field for testing the grid may be, for example, the field of a point mass, embedded a certain distance below a plane surface. It will be expressed, obviously, as a series of concentric circles of equal anomaly, with the maximum anomaly at the center. Deviations of the vertical for a given limited area of the field in such a case may be directly calculated with adequate accuracy by the numerical integration method. It is convenient to so situate the anomaly field as to have its center of greatest anomalies situated on the grid within the area of the zones investigated, while the point at which the deviation of the vertical is being determined, is located wherever the deviation of the vertical under the effect of the disturbing point mass reaches its peak.

These requirements will be satisfied if the depth a at which the point mass is situated is taken to equal the distance l_0 from the point at which the deviation of the vertical is being determined to the center of the anomaly field. The deviation of the vertical at point B (fig. 1) will then tend toward a maximum. To obtain results which serve to describe the properties of the grid as completely as possible, it is desirable to proceed part by part, first, for example, within the limits of radii from 0 to 5 km, then from 0 to 105 km; then from 0 to 329 km; and finally, from 0 to 1294 km, creating an anomaly field for each of these areas in accordance with the requirements above. It is convenient to adopt the following relationships between the dimensions of the model:

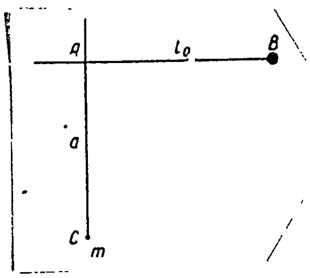

Fig. 1.

$$l_0 : a : R_0 = 1 : 1 : \frac{5}{3}, \qquad (23)$$

where R_0 designates the limiting radius of the portion of the grid involved. It is obvious that, in the light of (23), there is no need to calculate separately the anomaly field of the point mass for each of the areas of the grid. It is enough to perform the calculation once for one of the areas selected. For other areas of the grid, the anomaly field may be found by varying the scale of the model and the peak value of the field $\frac{fm}{a^2}$ (see formulas 25, 26 and 27 below). The effect of each of areas selected on the deviation of the vertical depending on the peak value of the anomaly $\frac{fm}{a^2}$ may be determined exactly (see formulas 32 and 33).

The Anomaly Field of a Point Mass

If the point mass m is situated at point C (Fig. 2), the distance of point M, at which the value of anomaly Δg is being determined, from the center A of the anomaly field equals ρ. The distance MC equals y. The angle formed by the line MC and a perpendicular is designated as β. The anomaly at an arbitrary point M of the field may then be expressed as :

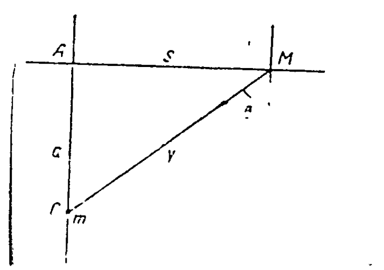

Fig. 2.

or

(24)

$$\Delta g = \frac{fm}{y^2} \cos\beta,$$

pr

$$\Delta g = \frac{fma}{(a^2+\rho^2)^{3/2}} = \frac{fm}{a^2}\frac{1}{\left(1+\frac{\rho^2}{a^2}\right)^{3/2}}. \qquad (24)$$

(24)

The quantity $\frac{fm}{a^2}$ will, obviously, be the maximum possible anomaly of that field. For convenience in calculation, let us solve the formula relative to ρ. This will make it possible to calculate ρ with ease for given values of Δg, varying by intervals of 10 or 5 mgls :

$$\rho = \frac{a\sqrt{1-\left(\frac{\Delta g}{fm}a^2\right)^{2/3}}}{\left(\frac{\Delta g}{fm}a^2\right)^{1/3}}. \qquad (25)$$

Positing

$$\sqrt[3]{\frac{\Delta g}{fm}a^2} = \cos\alpha, \qquad (26)$$

we obtain a working formula for computing the anomaly field:

$$\rho = a \operatorname{tg} \alpha. \qquad (27)$$

For given values of a and $\frac{fm}{a^2}$, the radii of isanomalic lines for chosen values of Δg may be obtained rather easily by means of this formula.

For checking the grid within the 0 - 5km area (R_o = 5 km), we have taken, in accordance with the condition set forth in (23), $a = l_o$ = 3 km, with $\frac{fm}{a^2}$ = 100 mgls. Table 2 gives the radii of circumferences of equal anomaly (every 5 mgls), computed from formulae 25, 26 and 27.

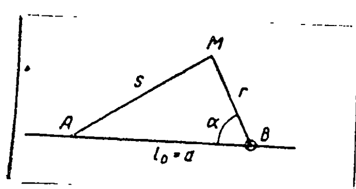

Fig. 3

Theoretical values for the deviation of the vertical for a limited area $0 \leqslant r \leqslant r_o$ may be obtained by subtracting the effect of the anomaly field outside of the circle of radius r_o from the total value for the deviation of the vertical. This effect will be small, since anomalies subside rapidly with removal from the center of the anomaly field, the Wening-Meines function decreasing at the same time.

Let us find the deviation of a vertical line $\Delta\xi''$ for the area $r_o < r <$ ... e have:

$$\Delta g = \frac{fm}{a^2}\frac{a^3}{(a^2+\rho^2)^{3/2}}.$$

It follo...

$$\rho^2 = l_0^2 + r^2 - 2rl_0\cos\alpha,$$

whence

$$\Delta g = \frac{fm}{a^2}\frac{1}{(2+x^2-2x\cos\alpha)^{3/2}}. \qquad (28)$$

(28)

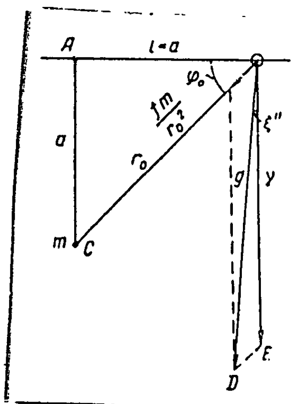

Fig. 4

Table 2

Δg	ρ, км	Δg	ρ, км
100	0,00	45	2,52
95	0,56	40	2,75
90	0,81	35	3,02
85	1,02	30	3,33
80	1,20	25	3,70
75	1,38	20	4,16
70	1,55	15	4,78
65	1,73	10	5,72
60	1,91	5	7,57
55	2,10	2	10,64
50	2,30		

where $x = \frac{r}{a}$.

The formula determining the deviation of a vertical line for a flat anomaly field has the general form:

$$\Delta\xi'' = -\frac{\varkappa''}{2\pi\gamma}\int_{r_0}^{\infty}\frac{\Delta g}{r}\cos\alpha\,dr\,d\alpha.$$

Substituting the expression for Δg from (28) into it and replacing r and r_0 respectively by x and x_0 (in accordance with the condition $x_0 = \frac{r_0}{a} = 2$), we find

$$\Delta\xi'' = -\frac{\varkappa''}{\pi\gamma}\cdot\frac{fm}{a^2}\int_{x_0}^{\infty}\frac{dx}{x}\int_0^{\pi}\frac{\cos\alpha\,d\alpha}{(2+x^2-2x\cos\alpha)^{3/2}}. \qquad (29)$$

By replacing α with π − 2β this integral is reduced to ellipt...

$$\int_0^{\pi}\frac{\cos\alpha\,d\alpha}{(2+x^2-2x\cos\alpha)^{3/2}} = -\frac{2}{[1+(1+x)^2]^{3/2}}\int_0^{\frac{\pi}{2}}\frac{\cos 2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}},$$

where k^2 ...

$$\int_0^{\frac{\pi}{2}}\frac{\cos 2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}} = \int_0^{\frac{\pi}{2}}\frac{\cos^2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}} - \int_0^{\frac{\pi}{2}}\frac{\sin^2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}}. \qquad (30)$$

Each of the two last integrals may be reduced to tabular elliptical integrals K and E in the following manner:

$$I_1 = \int_0^{\frac{\pi}{2}}\frac{\cos^2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}} = \frac{K-E}{k^2},$$

$$I_2 = \int_0^{\frac{\pi}{2}}\frac{\sin^2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}} = \frac{K}{k'^2} - \frac{K-E}{k^2k'^2},$$

$$I_1 - I_2 = \frac{1}{k^2}\left[2K - E\left(1+\frac{1}{k'^2}\right)\right]. \qquad (31)$$

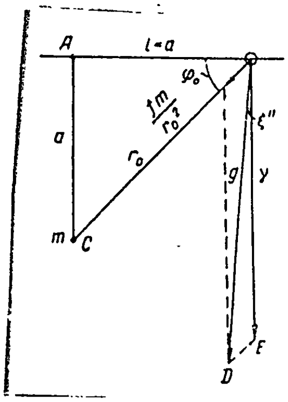

Fig. 4

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Substituting the expression for Δg from (28) into it and replacing r and r_0 respectively by x and x_0 (in accordance with the condition $x_0 = \frac{r_0}{a} = \frac{5}{3}$), we find

$$\Delta\xi'' = -\frac{\varkappa''}{\pi\gamma}\cdot\frac{fm}{a^2}\int_{x_0}^{\infty}\frac{dx}{x}\int_0^{\pi}\frac{\cos\alpha\,d\alpha}{(2+x^2-2x\cos\alpha)^{3/2}}. \qquad (29)$$

By replacing α with [illegible] this integral is reduced to ellip[illegible]

$$\int_0^{\pi}\frac{\cos\alpha\,d\alpha}{(2+x^2-2x\cos\alpha)^{3/2}} = -\frac{2}{[1+(1+x)^2]^{3/2}}\int_0^{\frac{\pi}{2}}\frac{\cos 2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}},$$

where k^2 [illegible]

$$\int_0^{\frac{\pi}{2}}\frac{\cos 2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}} = \int_0^{\frac{\pi}{2}}\frac{\cos^2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}} - \int_0^{\frac{\pi}{2}}\frac{\sin^2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}}. \qquad (30)$$

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$$I_1 = \int_0^{\frac{\pi}{2}}\frac{\cos^2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}} = \frac{K-E}{k^2},$$

$$I_2 = \int_0^{\frac{\pi}{2}}\frac{\sin^2\beta\,d\beta}{(1-k^2\sin^2\beta)^{3/2}} = \frac{K}{k'^2} - \frac{K-E}{k^2k'^2},$$

$$I_1 - I_2 = \frac{1}{k^2}\left[2K - E\left(1+\frac{1}{k^{2'}}\right)\right]. \qquad (31)$$

Here $k'^2 = 1 - k^2$ while K and E designate the well-known elliptic integrals

$$K = \int_0^{\frac{\pi}{2}} \frac{d\beta}{\sqrt{1 - k^2 \sin^2 \beta}}; \quad E = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \beta}\, d\beta, \qquad (31')$$

for which tables of numerical values exist for the variable $k = \sin\alpha$.

Substituting values for J_1 and J_2 from (31) and (29),

$$\Delta\xi'' = -\frac{\varkappa''}{2\pi\gamma} \cdot \frac{fm}{a^2} \int_{x_0}^{\infty} \frac{dx}{x^2 [1 + (1 + x)^2]^{1/2}} \left[E\left(1 + \frac{1}{k'^2}\right) - 2K\right], \qquad (32)$$

where

$$k^2 = \frac{4x}{1 + (1 + x)^2}.$$

We then make a final substitution:

$$u = \frac{x_0}{x}, \; du = -\frac{x_0}{x^2} dx$$

for $x = x_0$, $u = 1$. For $x = \infty$, $u = 0$, we find that

$$\Delta\xi'' = -\frac{\varkappa''}{2\pi\gamma} \frac{fm}{a^2} \cdot \frac{1}{x_0} \int_0^1 \frac{du}{[1 + (1 + x)^2]^{1/2}} \left[E\left(1 + \frac{1}{k'^2}\right) - 2K\right]. \quad (32')$$

We have obtained the formula determining the value of the deviation of a vertical line for an area of anomaly from 5 km to ∞, valid only in the case when $a : l_0 : r_0 = 1 : 1 : 5/3$. For other relations between a, l_0 and r_0, this formula will, obviously, appear somewhat different. The value of the integral from formula (32) may be calculated through numerical integration by the Gauss formula with three or five ordinates. By taking $\frac{fm}{a^2}$ = 100mgls, we get $\Delta\xi''$ = - 0".716.

The effect of a disturbing point mass on the deviation of a vertical line at point B may be determined in the following manner.

In Fig. 4, BD = g represents the force of gravity observed resulting from the total effect of a homogeneous plane gravitational field and the attraction of the disturbing mass, BE = γ is the normal force of gravity of a homogeneous plane layer and $\angle$ DBE is the deviation ξ of a vertical line.

The triangle BDE gives us.

$$\sin\xi = -\frac{\Delta g_0}{\gamma} \cos(\varphi_0 + \xi) \approx -\frac{\Delta g_0}{\gamma} \cos\varphi_0 = -\frac{fm}{r_0^2} \frac{1}{\gamma} \cos\varphi_0 =$$
$$= -\frac{fm}{a^2} \cdot \frac{1}{\gamma} \frac{a^2}{(a^2 + l_0^2)} \cos\varphi_0 = -\frac{1}{\gamma} \frac{fm}{a^2} \frac{a^2}{(a^2 + l_0^2)} \frac{l_0}{\sqrt{a^2 + l_0^2}} =$$
$$= -\frac{1}{\gamma} \frac{fm}{a^2} \frac{a^3}{(2a^2)^{3/2}} = -\frac{1}{\gamma} \frac{fm}{a^2} \cdot \frac{1}{\sqrt{8}},$$

The latter expression is obtained on the basis of the equation $a = l_0$. Since ξ is small, we find

$$\xi'' = -\frac{\varkappa''}{\gamma} \frac{fm}{a^2} \frac{1}{\sqrt{8}}. \qquad (33)$$

By taking $\frac{fm}{a^2}$ = 100mgls, we find ξ'' = 7".434. The deviation of the vertical from the effect of the central 0 - 5 km area will amount to:

$$\xi''_{0-5} = -7'',434 + 0'',716 = -6'',718.$$

Table 3 gives results for the deviation of the vertical from grids I, II and III, calculated specifically for the central zone (i.e. from formulas II, III and IV):

Table 3

Exact value	Grid I	Grid II	Grid III
$\xi = -6''.72$	$-3''.17$	$-6''.91$	$-6''.73$

Grid I yields an entirely incorrect result, since it fails to take into account the distinctive anomalies of the portion of the field concerned. However, it is true that in the example given, the gradient is exaggerated over its real value.

To verify the calculation of the grid for 0 - 105 km, 0 - 329 km and 0 - 1294 km, wr have adopted the same model of a point mass and a plane anomaly field, with the only difference that $\frac{fm}{a^2}$ is assumed to be different. The relations of dimensions within the model is preserved, i.e. $a : l_o : r_o = 1 : 1 : 5/3$. For this reason, formulas (32) and (33) remain valid for this particular case, while the numerical result varies proportionately to the variation of $\frac{fm}{a^2}$.

For the 0 - 105 km area, we have taken $\frac{fm}{a^2} = 200$ mgls, whence it follows that $\xi'' = -13''.434$;

for the 0 - 329 km area, $\frac{fm}{a^2} = 400$ mgls, and, as result, $\xi'' = -26''.867$;

for the 0 - 1294 km area, $\frac{fm}{a^2} = 400$ mgls, and $\xi'' = -26''.867$.

The results of comparisons between theoretical deviations of the vertical and deviations computed with the aid of the grid were as follows (Table 4):

Table 4

Area	ξ'', calculated theoretically	ξ'', obtained with grid	Error $\delta\xi''$, resulting from grid
0—105,1 км	— 13",434	— 13",455	+ 0",021
0—329,0	— 26 ,867	— 26 ,853	— 0 ,014
0—1294,0	— 26 ,867	— 26 ,839	— 0 ,028

The divergences obtained to do not exceed 0.2% of the values determined, whence it may be concluded that grid calculations were performed correctly.

In addition, let us examine specific typical examples of a complex anomaly field that may be encountered in practice, and let us compute for them the deviations of the vertical caused by the central zone of radius $r_o = 5$ km. We will perform the calculations by means of grids designed after formulas II, III and IV, i.e. grids of relatively low, average and high accuracy. The comparison of results from the three grids allows the determination of the error to be expected from a given grid relative to the degree of anomaly of the field. Table 5 presents the results of these comparisons for a set of artificial gravimetric maps (1-9). We give both the individual values of deviations of the vertical obtained for the 9 different maps, and the differences between the results obtained by means of grid III, which yields exact results, and grids I and II. The lower line gives the mean quadratic values for these differences, which may be considered as mean standard errors of the determination of deviations of the vertical by means of grids I and II.

As is apparent from the table, grid I may be used only in the case of an even anomaly field. Grid II is entirely adequate for all practical purposes, especially since its coefficients are convenient.

In dubious cases of complex anomaly fields it is possible to use grid III.

Таблица 5

No. map	ξ Grid III	ξ Grid II	ξ Grid I	η Grid III	η Grid II	η Grid I	$\Delta\xi$ III—II	$\Delta\eta$ III—II	$\Delta\xi$ III—I	$\Delta\eta$ III—I
1	−0",07	−0",21	−0",53	+1",09	+1",21	+1",08	+0",14	−0",12	+0",46	+0",01
2	−0 ,59	−0 ,63	−0 ,78	−0 ,47	−0 ,38	−0 ,08	+0 ,04	−0 ,09	+0 ,19	−0 ,39
3	−0 ,24	−0 ,31	−0 ,68	−0 ,64	−0 ,54	−0 ,21	+0 ,07	−0 ,10	+0 ,44	−0 ,43
4	−1 ,21	−1 ,23	−0 ,91	+0 ,89	+0 ,86	+0 ,49	+0 ,02	+0 ,03	−0 ,30	+0 ,40
5	+1 ,20	+1 ,31	+1 ,46	−0 ,13	−0 ,18	−0 ,21	−0 ,11	+0 ,05	−0 ,26	+0 ,08
6	+0 ,16	+0 ,08	−0 ,11	+0 ,81	+0 ,57	+0 ,25	+0 ,08	+0 ,24	+0 ,27	+0 ,56
7	−0 ,14	−0 ,10	+0 ,14	+1 ,34	+1 ,16	+0 ,40	−0 ,04	+0 ,18	−0 ,28	+0 ,94
8	+0 ,64	+0 ,50	−0 ,08	+0 ,74	+0 ,84	+1 ,12	+0 ,14	−0 ,10	+0 ,72	−0 ,38
9	−1 ,01	−1 ,02	−1 ,17	−0 ,52	−0 ,47	−0 ,10	+0 ,01	−0 ,05	+0 ,16	−0 ,42
							±0",09	±0",13	±0",38	±0",47
							[redacted]	±0",11	[redacted]	±0",42

Mean　　Mean

Grid for Spherical Anomaly Field

To check the grid for a spherical field of anomaly, let us posit, by analogy with the case of a plane field, a homogeneous sphere containing a point mass at a depth a below the surface (Fig. 5). In such a case, it is obvious that isanomalic lines will be concentric circles, with the peak anomaly at their center.

Point B, at which the deviation of the vertical was determined, is taken at a spherical distance S_0 from the center of the field. The relation existing between the dimensions of the model is similar to that in the case of the "flat" grid and is as follows:

$$a : S_0 : R_0 = 1 : 1 : \frac{5}{3},$$

where R_0 designates the spherical radius of the circle bounding the anomaly field. In our case R_0 = 1000 km and, therefore, S_0 = 600 km.

The anomaly Δg_n at an arbitrary point of the sphere equals, as we know,

$$\Delta g_n = g_0 - \gamma_0, \qquad (34)$$

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where g_0 is the force of gravity applied to a level surface at the point involved, and γ_0 is the normal gravity resulting from all masses evenly distributed within the sphere as a homogeneous globe. The value of g_0 is determined thus:

$$g_0 = g + \delta g_n, \qquad (35)$$

where g is the force of gravity resulting from all masses and "measured" at a point of the sphere, and δg_n represents the applied to a level surface from the surface of the sphere.

We may write

$$g = \bar{\gamma} + \Delta g ,$$

where $\bar{\gamma}$ is the normal force of gravity, resulting from the effect of the sphere (without a disturbing point mass), and Δg is the force of gravity originating from the disturbing point mass (see Fig. 5).

Then $\Delta g_n = \bar{\gamma} + \Delta g + \delta g_n - \gamma_0,$

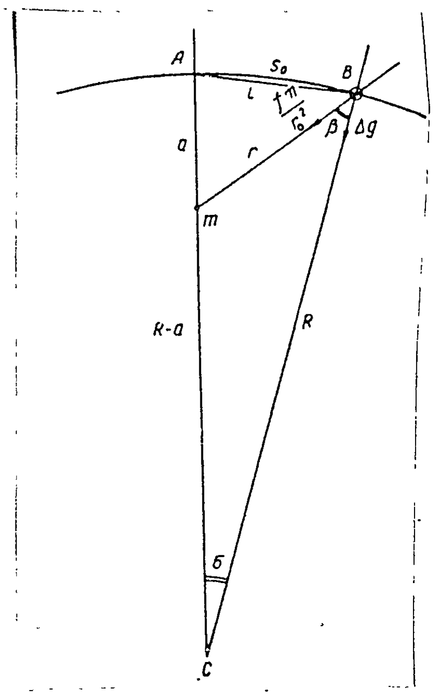

Fig. 5

while

$$\bar{\gamma} - \gamma_0 = -\frac{fm}{R^2},$$

and therefore

$$\Delta g_n \approx \Delta g + \delta g_n - \frac{fm}{R^2} \qquad (36)$$

or, if we eliminate the constant item $\frac{fm}{R^2}$, an operation which, as we know, will not affect the result of our determination of the deviation of the vertical, we may write:

$$\Delta g_n = \Delta g + \delta g_n. \qquad (36')$$

If we view a plane anomaly field as the limiting case of a spherical field ($R = \infty$), we see that, in that case, the two last terms of formula (36) will equal zero and, therefore, that the problem of the application of the force of gravity disappears.

For convenience in calculation, we will henceforth proceed as follows: we will resolve the field of anomalies Δg_n into two component fields, one caused solely by the action of the disturbing point mass, i.e. consisting entirely of values for Δg, the other composed solely of values for δg_n. We will calculate the effect on the deviation of the vertical for each of these two fields separately. The sum of all effects will, in accordance with (36), yield the actual value for the deviation of a vertical line at a given point B.

Determination of field of values for Δg.

From Fig. 5, we have:

$$r^2 = (R-a)^2 + R^2 - 2(R-a)R\cos\sigma;$$

$$\cos\beta = \frac{R-(R-a)\cos\sigma}{r}.$$

whence

$$\Delta g = \frac{fm}{r^2}\cos\beta = \frac{fm}{a^2}\,\frac{1+2\frac{R-a}{a}\sin^2\frac{\sigma}{2}}{\left[1+\frac{4R(R-a)}{a^2}\sin^2\frac{\sigma}{2}\right]^{3/2}}. \qquad (37)$$

Let us now put this expression for Δg in a form convenient for calculation. For this purpose, we may express anomalies in terms of the chord l of arc AB. Knowing the length of the chord, it is possible to find, with an accuracy sufficient for practical purposes, the length of the arc, i.e. the spherical distance between the point given on the sphere and the center of the field of anomalies. To do this, we may use the first two terms of the sine development of a small arc in series.

Thus, if $l = 2R\sin\frac{\sigma}{2}$, then

$$\Delta g = \frac{fm}{a^2}\,\frac{1+\frac{R-a}{2R^2a}l^2}{\left(1+\frac{R-a}{Ra^2}l^2\right)^{3/2}};$$

whence we get

$$l = \frac{a}{\sqrt{\frac{R-a}{R}}}\sqrt{\frac{\left(1+\frac{R-a}{2Ra}l^2\right)^{2/3}}{\left(\frac{\Delta g}{fm}a^2\right)^{2/3}} - 1}.$$

Let us introduce the auxiliary angle β:

$$\left(\frac{1+\frac{R-a}{2Ra}l^2}{\frac{\Delta g}{fm}a^2}\right)^{1/3} = \sec\beta;$$

then

$$l = \frac{a}{\sqrt{\frac{R-a}{R}}}\operatorname{tg}\beta,$$

while

$$1+\frac{R-a}{2R^2a}l^2 = 1+\frac{a}{2R}\operatorname{tg}^2\beta.$$

From the latter expression, we find l by successive approximations and, therefore, the spherical distance S by the approximate formula:

$$S = l + \frac{l^3}{24R^2} + \cdots$$

Let us now give the working formulas used in calculating the field of anomalies:

$$\frac{1}{\cos\beta_0}\left(1+\frac{a}{2R}\operatorname{tg}^2\beta_n\right)^{1/3} = \sec\beta_{n+1}, \qquad (38)$$

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whence we obtain β_{n+1}, which, once known, allows us to obtain:

$$l_{n+1}=\frac{a}{\sqrt{\frac{R-a}{R}}}\operatorname{tg}\beta_{n+1}, \qquad (39)$$

$$S=l+\frac{l^3}{24R^2}, \qquad (40)$$

$$l_x=S-\frac{S^3}{6R^2}. \qquad (40')$$

Here, l_x designates the projection of the spherical distance onto a plane tangential to the sphere at the center of the anomaly field. Table 6 gives the field of normal values for the force of gravity Δg originating from a point mass for a = 600 km and $\frac{fm}{a^2}$ = 400 mgl.

Table 6

Δg	$S_{км}$	Δg	$S_{км}$	Δg	$S_{км}$
400 мгл	0,0 км	250 мгл	390,9 км	100 мгл	80[illegible],8 км
390	83,8	240	410,8	90	860,0
380	119,8	230	431,1	80	917,6
370	148,0	220	452,0	70	985,0
360	173,1	210	473,5	60	106[illegible],0
350	195,7	200	495,7	50	1167,0
340	217,2	190	519,0	40	1301,0
330	237,7	180	543,4	30	1490,8
320	257,1	170	569,0	20	1805,5
310	296,4	160	596,0		
300	295,5	150	624,8		
290	314,5	140	655,6		
280	333,4	130	689,0		
270	352,3	120	725,5		
260	371,4	110	765,5		

We have the problem of calculate the deviation of the vertical caused by the anomaly field of a point mass. In doing this, we will circumscribe the area of anomalies considered by an outer radius of R_o. The calculation of the deviation of the vertical caused by such a field is best performed in the following manner: having found the total value of the deviation of the vertical caused by the point mass, we may subtract from it the deviation caused by the outer area, i.e. the anomaly field of the entire sphere, excluding the field bounded by the given circumference of the small circle. In this manner, the problem is much more easily solved than by the direct calculation of the deviation of the vertical caused by the effect of the circumscribed circular field.

Let us first calculate the effect of the anomaly field situated outside of the circle we have plotted. According to the Wening-Meines formula, the deviation of the vertical caused by a field of anomaly of radius ψ_0 may be expressed as:

$$\Delta\xi'' = \frac{\varkappa''}{2\pi\gamma}\int_{\psi_0}^{\pi}\int_{0}^{\pi} Q\,\Delta g\cos\alpha\,d\psi\,d\alpha. \qquad (41)$$

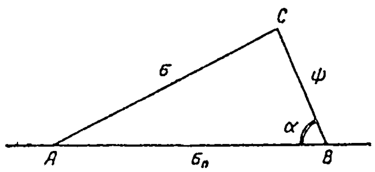

Fig. 6

But, from (37), we have

$$\Delta g = \frac{fm}{a^2}\frac{1+2\frac{R-a}{a}\sin^2\frac{\sigma}{2}}{\left[1+\frac{4(R-a)R}{a^2}\sin^2\frac{\sigma}{2}\right]^{3/2}},$$

while (Fig. 6):

$$\cos\sigma = \cos\psi\cos\sigma_0 + \sin\psi\sin\sigma_0\cos\alpha;\quad \sigma_0 = \frac{a}{R};\quad a = S_0.$$

Substituting Δg in (41), we find that

$$\Delta\xi'' = \frac{fm}{a^2}\frac{\varkappa''}{2\pi\gamma}\int_{\psi_0}^{\pi}\int_{0}^{\pi} Q\frac{1+2\frac{R-a}{a}\sin^2\frac{\sigma}{2}}{\left[1+\frac{4R(R-a)}{a^2}\sin^2\frac{\sigma}{2}\right]}\cos\alpha\,d\psi\,d\alpha. \qquad (42)$$

It is convenient to resolve this integral into two integrals, so as to convert these to tabular elliptical integrals following the transformation of the subintegral expression. It is not difficult to see that

$$\Delta\xi'' = \frac{fm}{a^2}\cdot\frac{\varkappa''}{2\pi\gamma}\left[\left(1-\frac{a}{2R}\right)\int_{\psi_0}^{\pi}\int_{0}^{\pi} Q\frac{\cos\alpha\,d\psi\,d\alpha}{\left[1+\frac{4R(R-a)}{a^2}\sin^2\frac{\sigma}{2}\right]^{3/2}} + \right.$$

$$\left. + \frac{a}{2R}\int_{\psi_0}^{\pi}\int_{0}^{\pi} Q\frac{\cos\alpha\,d\psi\,d\alpha}{\left[1+\frac{4(R-a)}{a^2}\sin^2\frac{\sigma}{2}\right]^{3/2}}\right], \qquad (43)$$

where

$$2\sin^2\frac{\sigma}{2} = 1 - \cos\psi\cos\sigma_0 - \sin\psi\sin\sigma_0\cos\alpha. \qquad (44)$$

But,

$$1+\frac{4R(R-a)}{a^2}\sin^2\frac{\sigma}{2} = \left(1+\frac{4R(R-a)}{a^2}\sin^2\frac{\psi+\sigma_0}{2}\right)\left\{1 - \frac{4R(R-a)\sin\psi\sin\sigma_0\cos^2\frac{\alpha}{2}}{a^2\left[1+\frac{4R(R-a)}{a^2}\sin^2\frac{\psi+\sigma_0}{2}\right]}\right\}. \qquad (45)$$

By introducing the expression

$$k^2 = \frac{4R(R-a)\sin\psi\sin\sigma_0}{a^2\left[1+\frac{4R(R-a)}{a^2}\sin^2\frac{\psi+\sigma_0}{2}\right]}, \qquad (46)$$

where $k^2 \leqslant 1$, we may posit

$$\alpha = \pi - 2\varphi,$$

whence

$$d\alpha = -2\,d\varphi,\quad \cos\alpha = -\cos 2\varphi; \qquad (47)$$

$$\cos\frac{\alpha}{2} = \sin\varphi;\quad \text{при}\ \alpha = 0,\quad \varphi = \frac{\pi}{2},\quad \alpha = \pi,\quad \varphi = 0.$$

After some fairly intricate transformations, we obtain

$$\Delta\xi'' = \frac{\varkappa'' fm}{\pi\gamma a^2}\left[\left(1-\frac{a}{2R}\right)\int_{\psi_0}^{\pi} Q\frac{\left[E\left(1+\frac{1}{k'^2}\right)-2K\right]d\psi}{k^2\left[1+\frac{4(R-a)R}{a^2}\sin^2\frac{\psi+\sigma_0}{2}\right]^{1/2}} + \right.$$

$$\left. + \frac{a}{2R}\int_{\psi_0}^{\pi} Q\frac{\left[K(1+k'^2)-2E\right]d\psi}{k^2\left[1+\frac{4R(R-a)}{a^2}\sin^2\frac{\psi+\sigma_0}{2}\right]^{1/2}}\right]. \qquad (48)$$

where

$$k^2 = \frac{\frac{4R(R-a)}{a^2}\sin\psi\sin\sigma_0}{1+\frac{4R(R-a)}{a^2}\sin^2\frac{\psi+\sigma_0}{2}}, \quad k'^2 = 1-k^2,$$

and K, E are complete elliptical integrals I and II of the type of (31), whose values are determined from parameter k from Janke and Emde's tables of elliptical integrals.

In calculating the value of $\Delta\xi''$ from (48), numerical integration by the Gauss method for $\psi_0 = \frac{R_0}{R} \approx \frac{1000}{6000} = 1/6$ and a = 600 km, $\sigma_0 = \frac{a}{R} \approx 1/10$ will provide the deviation of the vertical due to the effect of the anomaly field within a circle of radius R_0 = 1000 km, consisting entirely of Δg (normal values of the force of attraction of a point mass in points of the sphere):

$$\xi'' = -4''.633$$

Calculation of the Field of Reduction δg_n of Gravity in the Open Air

We may define the reduction of the force of gravity in open air δg_n as

$$\delta g_n = -kN_n, \qquad (49)$$

where N_n is the height of the displaced level surface above the sphere, and k is the coefficient for open air reduction for the height. The numerical value is k = 0.3086 if N_n is expressed in meters.

As we know, the displacement of the level surface away from the reference sphere is determined from Bruhn's formula with the aid of the potential of the disturbing point mass v_n:

$$N_n = \frac{v_n}{\gamma}$$

In our case

$$v_n = \frac{fM}{R} + \frac{fm}{r} - \frac{fM_0}{R} = \frac{fm}{r} - \frac{fm}{R},$$

where M is the mass of the sphere, m is the value of the point mass and M_0 is total mass $M_0 = M + m$. Within a constant margin of error (which will not affect the results of the calculation of the deviation of verticals), we may, therefore write $v_n = \frac{fm}{r}$.

Thus, for δg_n, we will obtain:

$$\delta g_n = -\frac{k}{\gamma}\frac{fm}{r}.$$

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By reference to Fig. 7:

$$r = [(R-a)^2 + R^2 - 2(R-a)R\cos\sigma]^{1/2}.$$

Therefore

$$\frac{fm}{r} = \frac{fm}{[(R-a)^2 + R^2 - 2(R-a)R\cos\sigma]^{1/2}} \text{ и}$$

$$\frac{k}{\gamma}\frac{fm}{r} = \frac{ka}{\gamma}\frac{fm}{a^2}\frac{1}{\left[1+\frac{R-a}{Ra^2}l^2\right]^{1/2}}, \qquad (50)$$

where l is the chord of arc σ.

Designating $\frac{ka}{\gamma}\frac{fm}{a^2}$ = L, we may write it in the following more usable form:

$$\delta g_n = -\frac{L}{\left[1+\frac{R-a}{Ra^2}l^2\right]^{1/2}}. \qquad (50')$$

It follows that $l = \sqrt{\frac{\left(\frac{L}{\delta g_n}\right)^2 - 1}{\frac{R-a}{Ra^2}}}$, and positing $\sec\alpha = \frac{L}{\delta g_n}$ we get:

$$l = \frac{a\,\mathrm{tg}\,\alpha}{\sqrt{\frac{R-a}{R}}}. \qquad (51)$$

In our case, $\frac{fm}{a^2}$ = 400 mgl, γ = 981,000 mgl, a = 600,000 m and, thus, L = 75.500 mgls.

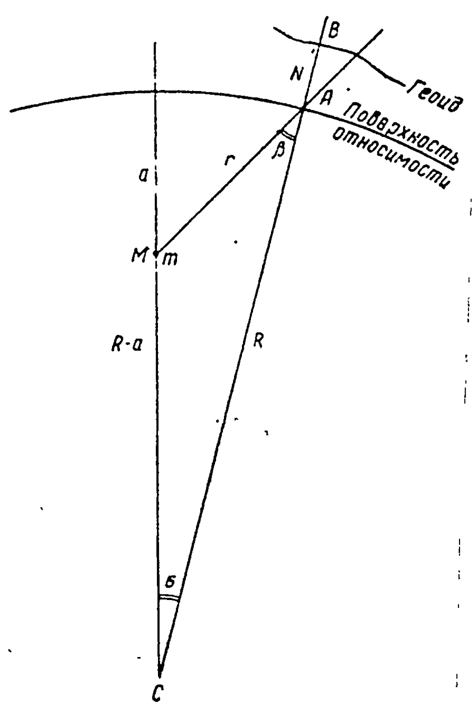

Fig. 7

As before (see formula 40'), we may project the anomaly field provided by formulas (51) onto a plane perpendicular to a radius leading from the center of the sphere to the center of the anomaly field. Then the projection l_x of the radius of the circle of isanomaly will be determined as follows:

$$l_x = l\cos\frac{\sigma_0}{2} = l\sqrt{1-\left(\frac{l}{2R}\right)^2} \approx l\left(1-\frac{l^2}{8R^2}\right). \qquad (52)$$

The results of calculations of l_x from (52) and (50) are presented in Table 7.

Table 7

δg_n	l_x	δg_n	l_x
−75,5 mgl	0,0	−50 mgl	712,2
−75	72,9	−45	847,4
−70	254,7	−40	1006,0
−65	372,3	−35	1200
−60	481,2	−30	1446
−55	592,2	−25	1778

To calculate the deviation of the vertical $\delta\xi''$ due to the reduction field in the open air, we may use the well-known Wening-Meines formula:

$$\delta\xi'' = \frac{\varkappa''}{2\pi\gamma_0}\int_{\psi_0}^{\pi}\int_0^{\pi} Q\,\delta g_n \cos\alpha\, d\psi\, d\alpha \qquad (53)$$

in which we will substitute δg_n with the expression in (50) or

$$\delta g_n = -\frac{ka}{\gamma_0}\frac{fm}{a^2}\frac{1}{\left[1+\frac{4(R-a)R}{a^2}\sin^2\frac{\sigma}{2}\right]^{3/2}}; \qquad (54)$$

then $\delta\xi''$ may be written as follows:

$$\delta\xi'' = -\frac{\varkappa'' ka}{2\pi\gamma_0^2}\frac{fm}{a^2}\int_{\psi_0}^{\pi}\int_0^{\pi} Q\frac{\cos\alpha}{\left[1+\frac{4(R-a)R}{a^2}\sin^2\frac{\sigma}{2}\right]^{3/2}}\, d\psi\, d\alpha =$$

$$= -\frac{\varkappa'' ka}{2\pi\gamma_0^2}\frac{fm}{a^2}\int_{\psi_0}^{\pi} Q\frac{[K(1+k'^2)-2E]\, d\psi}{\left[1+\frac{4R(R-a)}{a^2}\sin^2\frac{\psi+\sigma_0}{2}\right]^{3/2}}. \qquad (55)$$

The right-hand side of expression (53) has been transformed in the same manner as in the derivation of formula (48). In our case, a = 600 km

$$\psi_0 = \frac{R_0}{R} = \frac{1000\ км}{6000\ км} = \frac{1}{6}, \quad \gamma_0 = 981\,000 \text{ mgls.}$$

For these values, we obtain from formula (53)

$$\delta\xi'' = +1''.706.$$

Let us now calculate the total value of the deviation of the vertical due to the disturbing point mass. Let us designate the attraction of the point mass at point B, equal to $\frac{fm}{r^2}$, as v. The total deviation of a vertical line under the effect of the point mass will be expressed by means of the angle $gB\gamma = \xi$ (Fig. 8), which will be found from the expression:

$$\sin \xi = -\frac{v}{\gamma} \sin (\beta - \xi).$$

Since the angle of ξ is small, we may write quite accurately:

$$\xi = -\frac{fm}{r_0^2} \frac{1}{\gamma} \sin \beta.$$

It further follows from triangle CMB

$$\sin \beta = \sin \sigma_0 \frac{R-a}{r_0}; \quad r_0^2 = (R-a)^2 + R^2 - 2(R-a) R \cos \sigma_0.$$

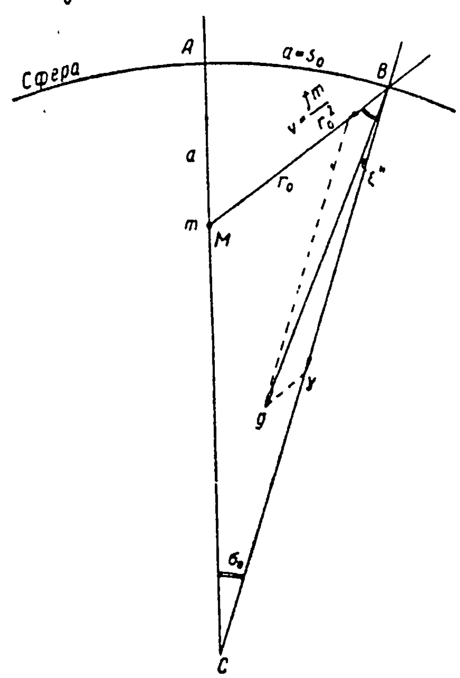

Fig. 8

Then

$$\xi = -\frac{fm}{\gamma} \cdot \frac{R - a \sin \sigma_0}{[(R-a)^2 + R^2 - 2(R-a) R \cos \sigma_0]^{3/2}} =$$

$$= -\frac{fm}{a^2} \frac{1}{\gamma} \frac{\frac{R-a}{a} \sin \sigma_0}{\left[1 + 4 \frac{R-a}{a} \frac{R}{a} \sin^2 \frac{\sigma}{2}\right]^{3/2}}.$$

Finally, expressing in angular units we have the following expression for the total deviation of the vertical due to the effect of the disturbing point mass:

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$$\xi'' = -\frac{fm\,\varkappa''}{a^2\,\gamma}\,\frac{\frac{R-a}{a}\sin\sigma_0}{\left[1+4\frac{R-a}{a}\frac{R}{a}\sin^2\frac{s}{2}\right]^{3/2}} \qquad (56)$$

Positing that $\frac{fm}{a^2} = 400$ mgls, $a = 600$ km, $\sigma_0 = \frac{S_0}{R} = \frac{a}{R}$, we get $\xi'' = -28''.931$.

Thus,

$$\xi''_{0-1000\,km} = \xi'' - \Delta\xi'' - \delta\xi'' = -28''.931 + 4''.633 - 1''.706 = -26''.004.$$

This value must be corrected for the displacement of the mass center, which may be described by the following formula:

$$\delta = +\frac{\varkappa''\,x_0}{R}\sin\sigma.$$

In our case, $x_0 = 19.42$ m, $\sigma \approx \frac{600km}{6000km} = \frac{1}{10}$.

Then, $\sin\sigma_0 = 0.02877$ and $\delta = 0''.629\sin\sigma_0 = +0''.018$.

The result of a determination by means of the grid of the deviation of the vertical caused by the same spherical anomaly field with a spherical radius of 1000 km was found to be $-25''.934$. Thus, the error in the grid determination was $0''.05$. Such a result may be considered entirely satisfactory. For purposes of verification, the effect of zones XX and XXI was determined by means of th

$$\int_{\psi_1}^{\psi_2}\frac{d}{d\psi}S(\psi)\sin\psi\,d\psi.$$

III. Calculation of the Elevation ζ of the Quasi-Geoid for an Anomaly Field with a Radius $R_0 = 2000$ km by Means of Spherical Grid

As indicated earlier, the design of the grid for the computation of deviations of the vertical provides for convenience also in the calculation of elevations ζ of the quasi-geoid. Since the effect of anomalies on ζ does not depend on azimuth, the entire area taken into account is divided into equal sections (sectors). In addition, we have narrowed the width of the distant zones to some extent, in view of the fact that their effect is much more significant in determining ζ than in finding deviations ξ and η of the vertical. Therefore, in passing on to the calculation of the spherical grid for the determination of ζ we may retain all the zonal radii adopted in designing a spherical grid for the calculation of deviations of verticals (see Table 1).

Let us separate out a central zone ($r_0 = 5$ km), for which we may postulate the constant mean value for the Stokes function of $F(r) = 1$. The right-hand portion of formula (1) will

$$\zeta = \frac{1}{2\pi\gamma}\int_0^{r_0}\int_0^{2\pi}\Delta g(r,\alpha)\,dr\,d\alpha + \frac{1}{2\pi\gamma}\int_{r_0}^{R_0}\int_0^{2\pi}\Delta g(r,\alpha)\,F(r)\,dr\,d\alpha.$$

The first term $\frac{1}{2\pi\gamma}\int_0^{r_0}\int_0^{2\pi}\Delta g(r,\alpha)\,dr\,d\alpha$, allowing for the effect of the central zone, may be easily calculated, if we admit that the anomalies $\Delta g(r,\alpha)$ vary in a linear manner along the radii or that ;

$$\Delta g(r, \alpha) = \Delta g_0 + Ar, \qquad (57)$$

where $A = \frac{\partial \Delta g}{\partial r}$ is the horizontal gradient of anomalies which we assume is constant within the limits of the central zone along each radius, and Δg_0 is the value of the anomaly at the central point for which ζ is being determined.

Substituting (57) into the [illegible] we obtain

$$\Delta\zeta_0 = \frac{1}{2\pi\gamma}\int_0^{r_0}\int_0^{2\pi}(\Delta g_0 + Ar)\,dr\,d\alpha.$$

Following the integration along r within the limits 0 to r_0, which may here be performed directly, we have

$$\Delta\zeta_0 = \frac{1}{2\pi\gamma}\int_0^{2\pi}\left(\Delta g_0 r_0 + \frac{Ar_0^2}{2}\right)d\alpha.$$

Through partial further integration and simplification, we get

$$\Delta\zeta_0 = \frac{r_0}{\gamma}\Delta g_0 + \frac{r_0}{4\pi\gamma}\int_0^{2\pi} A r_0\,d\alpha.$$

Substituting here from (57) $Ar_0 = \Delta g(r_0, \alpha) - \Delta g_0$, where $\Delta g(r_0, \alpha)$ is the value of the anomaly at the boundary of the cer[illegible]

$$\Delta\zeta_0 = \frac{r_0}{\gamma}\Delta g_0 + \frac{r_0}{4\pi\gamma}\int_0^{2\pi}[\Delta g(r_0, \alpha) - \Delta g_0]\,d\alpha$$

or, after integration and simplification,

$$\Delta\zeta_0 = \frac{r_0}{2\gamma}\Delta g_0 + \frac{r_0}{4\pi\gamma}\int_0^{2\pi}\Delta g(r_0, \alpha)\,d\alpha.$$

Let us represent the remaining integral in simplified form as the sum:

$$\Delta\zeta_0 = \frac{r_0}{2\gamma}\Delta g_0 + \frac{r_0}{4\pi\gamma}\Sigma\,\Delta g(r_0, \alpha)\,\Delta\alpha.$$

Taking $\Delta\alpha = \frac{2\pi}{m}$, where m is the number of sections, we find a final expression for the effect of the central zone on ζ:

$$\Delta\zeta_0 = \frac{r_0}{2\gamma}\Delta g_0 + \frac{r_0}{2m\gamma}\sum_{k=1}^{k=m}\Delta g(r_0, \alpha_k). \qquad (58)$$

We have taken the number of sections as m = 8 and r_0 = 5 km. For these numerical values, formula (58) may be written as:

$$\Delta\zeta_0 = 0{,}00255\,\Delta g_0 + 0{,}00032\sum_{k=1}^{k=8}\Delta g(r_0, \alpha_k). \qquad (59)$$

We may note that here Δg_0 is the anomaly of the central point, for which ζ is being determined; $\Delta g(r_0, \alpha_k)$ is the anomaly at the boundary of the central zone at points on a circumference with azimuths: 0, 45, 90, 135, 180, 225, 270, 315°. All Δgs are expressed in milligals.

To check the accuracy acheived in using this formula, let us use it calculate ζ for an anomaly field expressed

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by the following non-linear formula (particular case):

$$\Delta g = \frac{k}{a+r}. \qquad (60)$$

Here k and a are constant quantities, which we may find from the conditions $r = 0$, $\Delta g_0 = 100$ mgls, while for $r = r_0 = 5$ km, $\Delta g_I = 50$ mgls. Under these conditions, $k = 500$, $a = 5$, and the formula for the anomaly field becomes

$$\Delta g(r, \alpha) = \frac{500}{5+r}.$$

With these values for Δg, we find $\Delta\zeta_0$ for the central point from formula (59)

$$\Delta\zeta_0 = 0{,}00255 \cdot 100 + 0{,}00032 \cdot 400 = 0{,}383\ м, = 38{,}3\ см.$$

Calculations by means of the Stokes formula, in this instance, give $\Delta\zeta_0 = 35.3$ cm, i.e. the divergence amounts to 3 cm.

If we were to take a field of anomalies varying in a strictly linear manner along a radius, such as for example $\Delta g(r) = 100 + 20r$ (for $r = 0$, $\Delta g_0 = 100$ mgls, and for $r = 5$ km, $\Delta g(r_0) = 200$ mgls), calculations by means of formula (59) should yield a result closer to that found by solving the Stokes formula. Indeed, in the example cited, the Stokes formula gives $\Delta\zeta_0 = 0.7645$ m, while formula (59) yields $\Delta\zeta_0 = 0.767$. In cases when the variation of anomalies departs significantly from a linear pattern within the central zone of 5 km radius, more exact formulas should be used, which we give here without their derivation.

$$\Delta\zeta_0 = 0{,}000\,612\,\Delta g_0 + 0{,}000\,076\,f(r_1) + 0{,}000\,419\,f(r_2) + 0{,}000\,066\,f(r_3),$$

$$f(r_i) = \sum_{k=1}^{8} (g-\gamma)_{ik}. \qquad \text{(IIIa)}$$

$$\Delta\zeta_0 = 0{,}000\,10\,\Delta g_0 + 0{,}000\,057\,f(r_1) + 0{,}000\,098\,f(r_2) + 0{,}000\,116\,f(r_3) + 0{,}000\,098\,f(r_4) + 0{,}000\,048\,f(r_5),$$

$$f(r_i) = \sum_{k=1}^{12} (g-\gamma)_{ik}. \qquad \text{(IVa)}$$

For formula IIIa: $r_1 = 1.05$ km, $r_2 = 2.88$ km, $r_3 = 4.52$ km, and for formula IVa: $r_1 = 0.42$ km, $r_2 = 1.31$ km, $r_3 = 2.60$ km, $r_4 = 3.89$ km, $r_5 = 4.78$ km; $(g-\gamma)_{ik}$ represents the anomalies at point k of a circumference of radius r_i. The value of $\Delta\zeta_0$ is expressed in meters. Let us examine the effect of the area $5 < r < 1000$ km, w[illegible] designate as $\Delta\zeta_I$:

$$\Delta\zeta_I = \frac{1}{2\pi\gamma} \int_{r_0}^{R} \int_{0}^{2\pi} \Delta g(r, \alpha)\, F(r)\, dr\, d\alpha.$$

Let us represent the two integrals entering herein in the approximate form of the sum:

$$\Delta\zeta_i = \frac{1}{2\pi\gamma} \Sigma\Sigma\, \Delta g\,(r, \alpha)\, F\,(r)\, \Delta r\, \Delta\alpha.$$

To use this formula with a spherical grid, we proceed as follows. We have

$$\Delta r = r_{i+1} - r_i \;(i = 0, 1, 2, 3 \ldots 21);$$

We take the radii r_i from Table 1. $\Delta\alpha$ we consider as assuming the same values as those adopted in computing the spherical grid for the calculation of deviations of the vertical, namely: within the area from $r_0 = 5$ km to $r_8 = 102.5$ km, we posit $\Delta\alpha = \frac{2\pi}{16}$, and for the area from $r_8 = 102.5$ km to $R_0 = r_{21} = 1000$ km, $\Delta\alpha = \frac{2\pi}{24}$. For the Stokes function $F(r)$ in each zone we take a mean value, corresponding to the mean value of the radius of the zone $r_{cp} = \frac{r_n + r_{n+1}}{2}$. We find the value of the mean anomaly for each trapezium by averaging anomalies within the area of the trapezium. Keeping all this mind, we may write for $\Delta\zeta_1$ the following formula with numerical coefficients:

$$\begin{aligned}\Delta\zeta_i = [(15 \overset{16}{\Sigma} \Delta g_{I} + 22 \overset{16}{\Sigma} \Delta g_{II} + 32 \overset{16}{\Sigma} \Delta g_{III} + 47 \overset{16}{\Sigma} \Delta_{IY} + 69 \overset{16}{\Sigma} \Delta g_{Y} + \\ + 101 \overset{16}{\Sigma} \Delta g_{YI} + 149 \overset{16}{\Sigma} \Delta_{YII} + 219 \overset{16}{\Sigma} \Delta g_{YIII}) + (118 \overset{24}{\Sigma} \Delta g_{IX} + 148 \overset{24}{\Sigma} \Delta g_{X} + \\ + 186 \overset{24}{\Sigma} \Delta g_{XI} + 232 \overset{24}{\Sigma} \Delta g_{XII} + 288 \overset{24}{\Sigma} \Delta g_{XIII}) + (262 \overset{24}{\Sigma} \Delta g_{XIY} + 304 \overset{24}{\Sigma} \Delta g_{XY} + \\ + 353 \overset{24}{\Sigma} \Delta g_{XYI} + 408 \overset{24}{\Sigma} \Delta g_{XYII} + 467 \overset{24}{\Sigma} \Delta g_{XYIII} + 529 \overset{24}{\Sigma} \Delta g_{XIX} + \\ + 596 \overset{24}{\Sigma} \Delta g_{XX} + 668 \overset{24}{\Sigma} \Delta g_{XXI})] \cdot 10^{-5},\end{aligned} \tag{61}$$

where Δg with a subscript designates an average value of anomaly for each trapezium of a zone, the zone being indicated by the Roman numeral.

This formula has been tested by means of two examples of anomaly fields, expressed as functions of the radius r:

$$1)\ \Delta g = kr \text{ and } 2)\ \Delta g = \frac{k}{r}. \tag{62}$$

<u>Example 1</u> $\Delta g = kr$.

<u>Calculation</u> ζ for this value for anomalies, we find from formula (1)

$$\zeta = \frac{k}{2\pi\gamma} \int_0^{R_0}\int_0^{2\pi} rF\,(r)\, dr\, d\alpha = \frac{k}{\gamma}\int_0^{R_0} rF\,(r)\, dr. \tag{63}$$

Resolving the integral into three integrals according to the three areas of the grid, i.e. the zones bounded by radii $0 - r_8$, $r_8 - r_{13}$ and $r_{13} - r_{21}$ $(r_{21} = R_0)$, and assuming the mean values $F_I(r) = +1.04$ for the first area, $F_{II}(r) = +1.15$ for the second, and $F_{III}(r) = +1.22$ for the third, we may find the sum of all three integrals by direct integration, i.e.

$$\zeta = \frac{k}{2\gamma} [(F_I - F_{II})\, r_8^2 + (F_{II} - F_{III})\, r_{13}^2 + F_{III}\, r_{21}^2], \tag{64}$$

or, by introducing the numerical values given above for F,

$$\zeta = \frac{k}{2\gamma}(-0,11\, r_8^2 - 0,07\, r_{13}^2 + 1,22\, r_{21}^2).$$

Positing k = 1 and expressing r in kilometers, we obtain anomalies in milligals. Calculating ζ from formula (64), we find ζ = 617.8 m, while calculating from formulae (59) and (61), we get

$$\zeta = 617.7 \text{ m.}$$

The above example tests only the distant zones adequately, since for $\Delta g = kr$, large anomalies may occur only in them. To test the design of the grid especially for the closer zones from r_0 = 5 km to r_8 = 102.5 km, let us use a second example.

Example 2. $\Delta g = \frac{k}{r}$.

As in example 1, we calculate the value of ζ from formula (55)

$$\zeta = \frac{k}{\gamma}\int_{r_0}^{r_8}\frac{1}{r}F(r)\,dr = \frac{k}{\gamma}F_1(\ln r_8 - \ln r_0) = \frac{k}{\gamma}F_1 M(\lg r_8 - \lg r_0). \quad (65)$$

Here we must posit: r_0 = 5 km, r_8 = 102.6 km and F_1 = 1.04 (on the average). If we assume the value 100,000 for the coefficient k, and compute r in kilometers, the anomaly $\Delta g = \frac{k}{r}$ will be expressed in milligals, and ζ will be in meters. By formula (65), we find ζ = 319.7 m. By means of the grid, (59) and (61), we obtain ζ = 320 m.

The comparison of results obtained by different methods shows that grid calculations (59) and (61) were performed correctly.

For $\frac{1000}{R} < \psi < \frac{2000}{R}$ we have

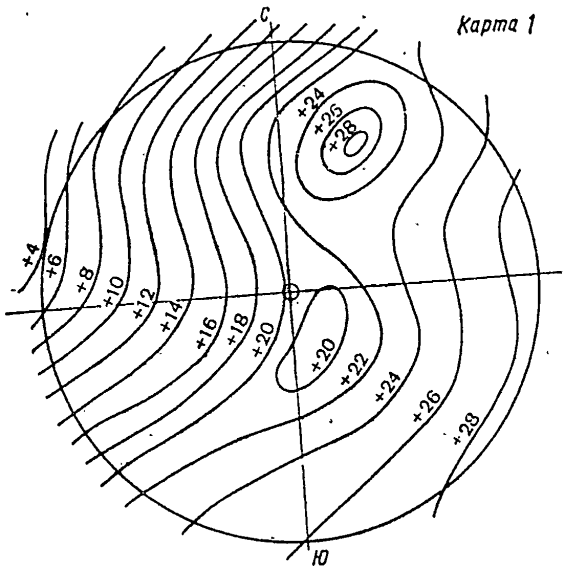
$$\Delta\zeta_2 = [424 \sum^{48} \Delta g_{XXII} + 462 \sum^{48} \Delta g_{XXIII} + 494 \sum^{48} \Delta g_{XXIV} + 518 \sum^{48} \Delta g_{XXV} + 528 \sum^{48} \Delta g_{XXVI}]\cdot 10^{-6}.$$

For purposes of verification, the values of coefficients for zones XXI and XX were calculated by means of the integral

$$\int_{\psi_1}^{\psi_2} S(\psi)\sin\psi\, d\psi.$$

The computation techniques using the formulas given are described in a paper by the author in Collected Papers on Geodesy, No 8, 1945.

Карта 1



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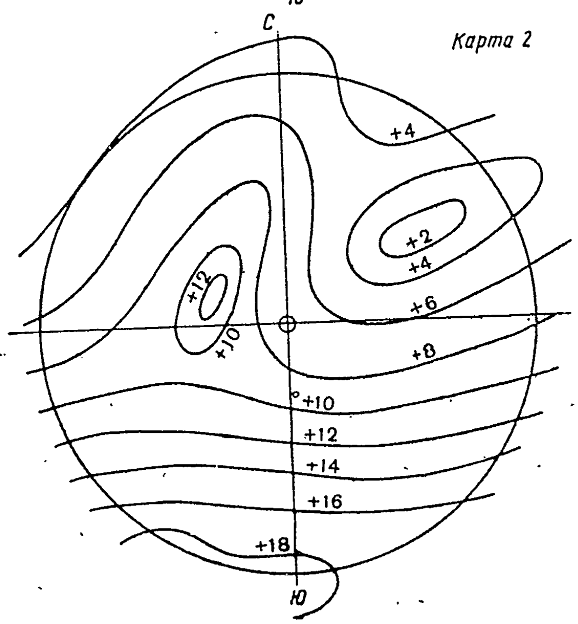

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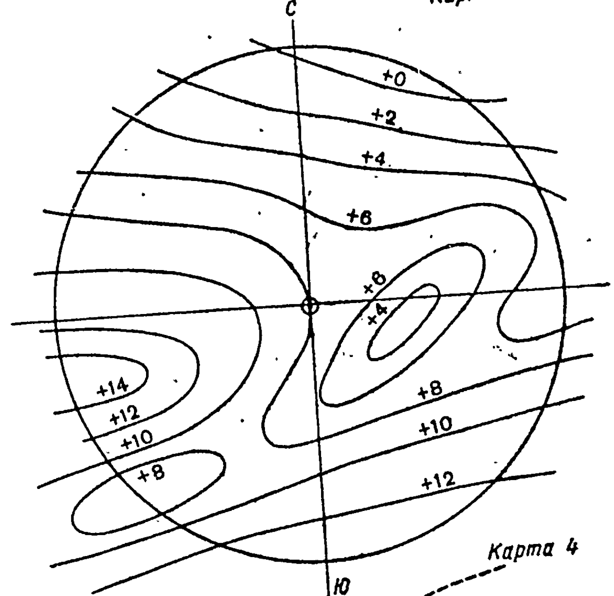

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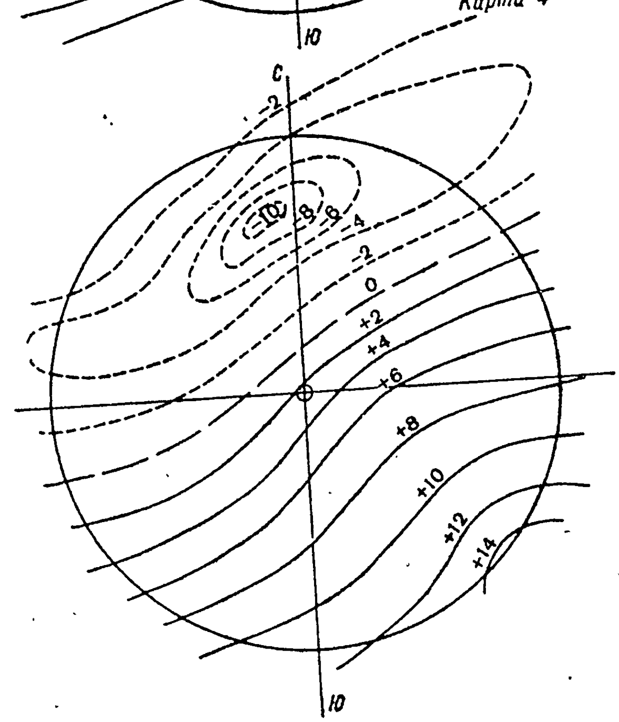

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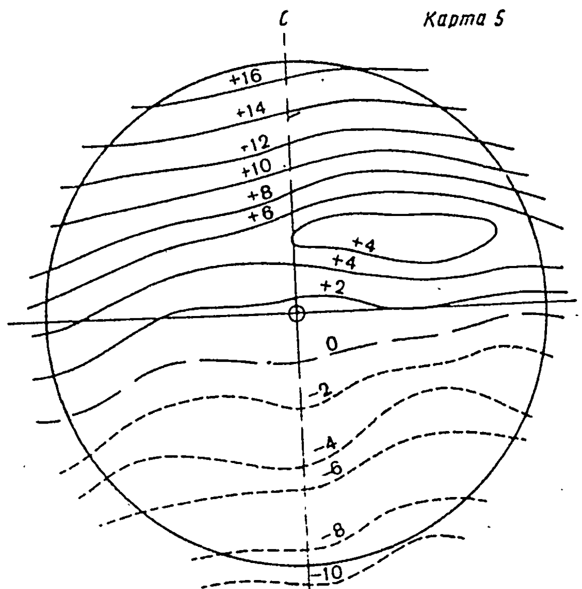

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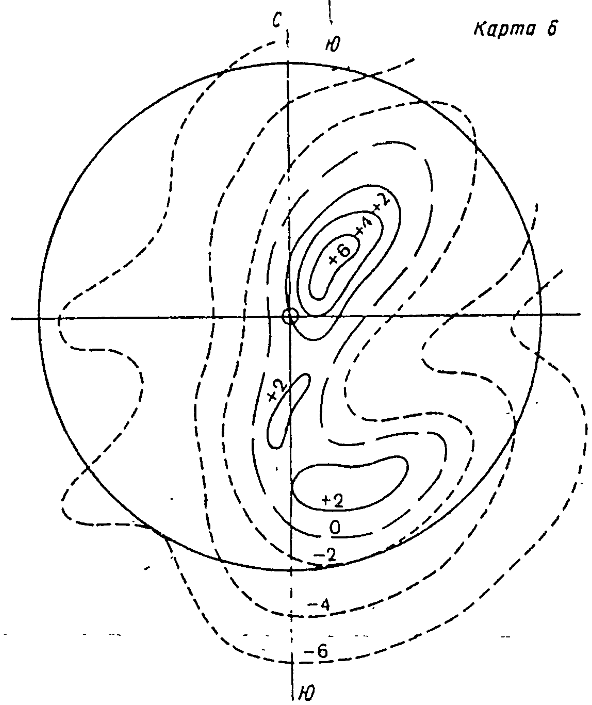

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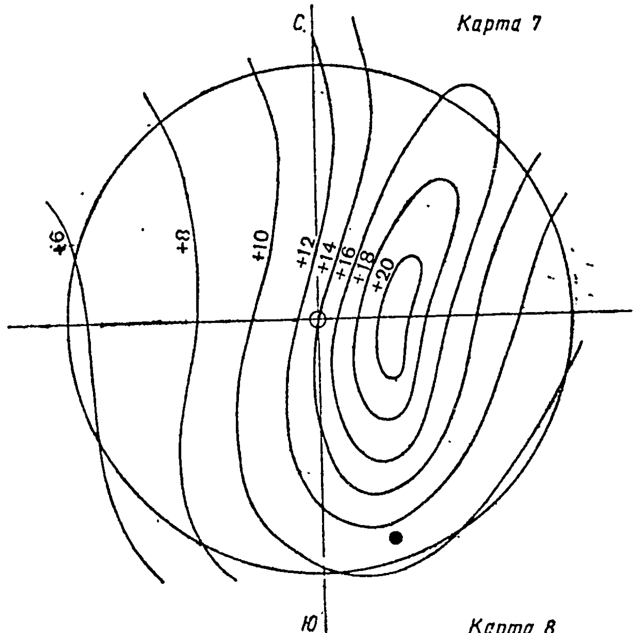
Карта 7



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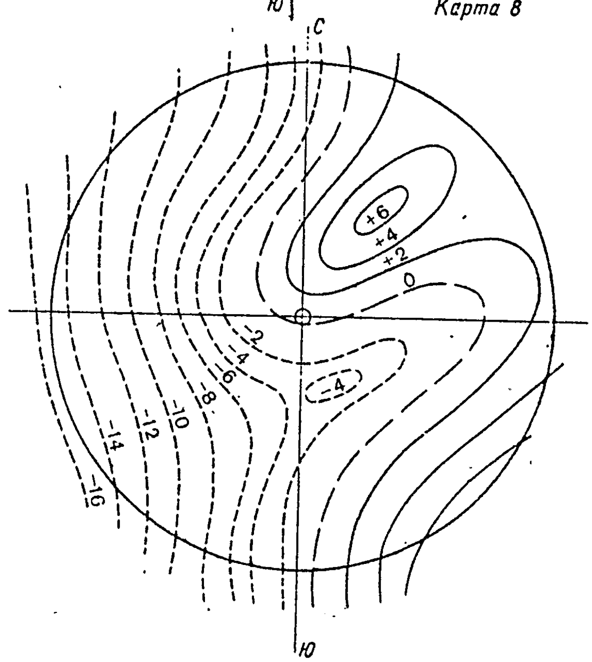
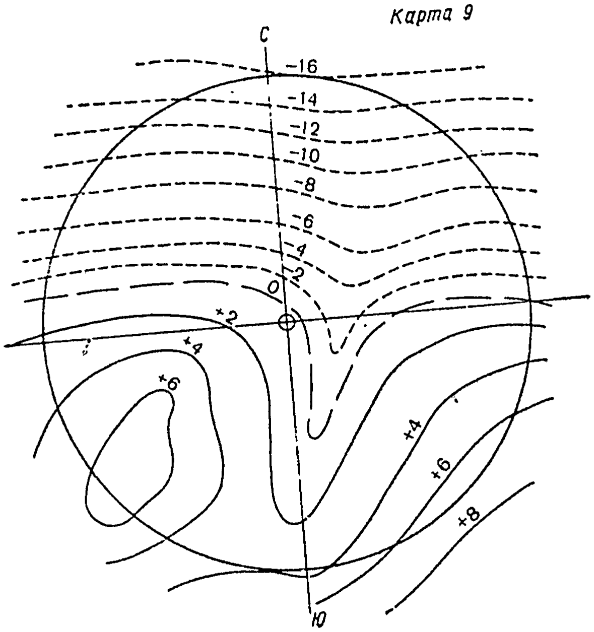


Карта 9



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FORMULAS AND TABLES FOR THE CALCULATION OF GEODETIC COORDINATES BY THE MOLODENSKIY METHOD

Works of the Central Scientific Research Institute of Geodesy, Aerial Surveying and Cartography, No 121, pp 77 - 104

V. F. Yeremeyev

Introduction

In 1952, M. S. Molodenskiy proposed a new method for the solution of basic geodetic problems [1]. Molodenskiy defines the position of a point in space by means of orthogonic curvilinear coordinates H, B and L, where H is the distance from the ellipsoid of revolution used as a basis for calculations, B is the geodetic latitude of the point, and L is geodetic longitude. The solution of geodetic problems by the method is given in general form in elementary functions. The formulas used differ from the formulae of the usual method, which make use of infinite series. Instead of curves on the ellipsoid (geodetic lines), Molodenskiy makes use of segments of a right line passing through the points under investigation*. The formulae for solving the direct and reverse geodetic problems by the Molodenskiy method have one and the same form regardless of the distance between the points. In the traditional method for solving these problems, the formulas corresponding to various distances differ and are presented in the form of infinite series, the general term of the series, as a rule, being unknown.

Molodenskiy replaces the spheroical triangles with plane ones. Instead of the geodetic azimuths of the geodetic lines, he determines the geodetic azimuths of perpendicular sections passing through the initial and terminal points. In practice, as we know, the latter are dealt with in the measurement of horizontal angles in triangulation.

This paper provides practical formulae and models of tables for solving the direct and reverse geodetic problems, formulae for calculating the reductions of spheroidic angles to plane ones, and formulae for calculating the sides of plane triangles. All formulae have been checked by means of examples; in the order in which it is done in practice in processing a triangulation, a triangle, all of whose summits are situated on the ellipsoid, has been solved by the Molodenskiy method.

Henceforth, the following symbols will be used:

a, b : respectively, the long and short semi-axis of the reference ellipsoid;

$B_n L_n$: geodetic coordinates of point n;

$\Delta B_{12} = B_2 - B_1$: difference of geodetic latitudes;

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*The following article in this collection sets forth an uncomplicated technique for proceeding from the chord to the geodetic line.

$L_{12} = L_2 - L_1$: difference of geodetic longitudes

$B_m = \frac{B_1 + B_2}{2}$: mean latitude

$S_{m,n}$: length of chord between points m and n;

$\psi_{m,n}$: spherical angular distance between points m and n;

$A_{m,n}$: geodetic azimuth of section perpendicular at point m, containing point n;

$z_{m,n}$: zenithal distance of chord $S_{m,n}$ at point m;

$\alpha_{m,n}$: azimuth of chord $S_{m,n}$ on sphere;

$n_{m,n}$: cosine of angle at point m of chord $S_{m,n}$ and and a right line parallel to the axis of rotation of the Earth;

$M_m N_m$: radii of curvature of surface of reference ellipsoid, respectively in the direction of the geodetic meridian and the first vertical at the point having coordinates B_m and L_m;

A_n: spheroidal angle at point n,

α_n: plane angle (angle formed by chords) at point n;

R_m: mean radius of curvature of the surface of the reference ellipsoid at point m;

ε: spherical residue of triangle;

e: eccentricity of ellipsoid.

1. Formulae for the Solution of the Reverse Geodetic Problem

Problem: given known geodetic coordinates B_1, L_1 and B_2, L_2 of two points situated on the ellipsoid (the dimensions of the ellipsoid are known), it is required to find the distance S_{12} between them (i.e. to find the length of a chord) and the geodetic azimuths A_{12} and A_{21} of direct and reverse perpednicular sections, passing through the points given. Thus, B_1, L_1, B_2 and L_2 are given, and we are required to find S_{12}, A_{12} and A_{21}.

After certain transformations of formulae (18) and (19) of paper [1] we obtain the following practical formulae for determining the length S_{12} of the chord:

$$\left(\frac{S_{12}}{N_1}\right)^2 = 4\frac{N_2}{N_1}\sin^2\frac{\psi}{2} - k_0\left(\frac{N_2}{N_1}\sin B_2 - \sin B_1\right)^2 + \left(\frac{N_2}{N_1} - 1\right)^2, \quad (1)$$

where

$$\sin^2\frac{\psi}{2} = \sin^2\frac{\Delta B_{12}}{2} + \cos B_1 \cos B_2 \sin^2\frac{\Delta L_{12}}{2}.$$

$$k_0 = e^2(2 - e^2). \quad (2)$$

In Krasovskiy's ellipsoid, $K_0 = 0.1334\ 2041 \cdot 10$.

From formulae (31, 32, 41 and 42) of [1], we obtain a group of formulae for the calculation of geodetic azimuths:

$$\sin(A_{21} - \alpha_{21}) = \sin A_{21} \sin \alpha_{21} \frac{\cos B_2}{\cos B_1 \sin \Delta L_{12}} \times$$

$$\times e^2\left(\frac{N_2}{N_1}\sin B_2 - \sin B_1\right), \quad (3)$$

$$\operatorname{ctg} \alpha_{21} = \frac{\sin \Delta B_{12}}{\cos B_1 \sin \Delta L_{12}} - \sin B_2 \operatorname{tg}\frac{1}{2}\Delta L_{12}, \quad (4)$$

$$A_{21} = (A_{21} - \alpha_{21}) + \alpha_{21}, \quad (5)$$

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$$\operatorname{ctg}\frac{1}{2}(\alpha_{12}-\alpha_{21})=\frac{\sin B_m}{\cos\frac{1}{2}\Delta B_{12}}\cdot\operatorname{tg}\frac{1}{2}\Delta L_{12}, \tag{6}$$

$$A_{12}\approx A_{21}+(\alpha_{12}-\alpha_{21}). \tag{7}$$

The later relation involves an accuracy to 0".001 for distances up to 1000 km.

To verify the calculations of geodetic azimuths A_{12} and A_{21}, we may use the adequately accurate formula:

$$\sin A_{12}\cos B_1+\sin A_{21}\cos B_2\approx\sin(A_{12}-\alpha_{12})\,[\cos\alpha_{12}\cos B_1+ +\cos\alpha_{21}\cos B_2], \tag{8}$$

or

$$\sin A_{12}\cos B_1+\sin A_{21}\cos B_2\approx\sin(A_{12}-\alpha_{12})\,[\cos A_{12}\cos B_1+ +\cos A_{21}\cos B_2]. \tag{9}$$

These formulae may be obtained by using the relations

$$\sin\alpha_{12}\cos B_1+\sin\alpha_{21}\cos B_2=0 \text{ и } A_{12}-\alpha_{12}\approx A_{21}-\alpha_{21}.$$

Geodetic azimuths for distances of over 1000 km may be calculated from formulae (3), (4) and (5) and the relations:

$$\sin(A_{12}-\alpha_{12})=\sin A_{12}\sin\alpha_{12}\frac{\cos B_1}{\cos B_2\sin\Delta L_{12}}\,e^2\left(\sin B_2-\frac{N_1}{N_2}\sin B_1\right), \tag{10}$$

$$\operatorname{ctg}\alpha_{12}=\frac{\sin\Delta B_{12}}{\cos B_1\sin\Delta L_{12}}+\sin B_1\operatorname{tg}\frac{1}{2}\Delta L_{12}, \tag{11}$$

$$A_{12}=(A_{12}-\alpha_{12})+\alpha_{12}. \tag{12}$$

For a general and combined verification of calculations of the distance S_{12} and geodetic azimuths A_{12} and A_{21} we may use the relations (21) or (22) and (23) below and formulas (19) and (20). In doing this for small distances ($S_{12}<100$ km), one approximation is sufficient in calculating n_{12}, while for a distance of 100 km $<S_{12}<$ 1000 km, two approximations are required, and for distances of $S_{12}>$ 1000 km three or perhaps more are necessary to check thousandths of a second in the azimuths and the eighth significant figure in the distance (see examples of calculations). For lower degrees of accuracy, fewer approximations are needed.

Calculations in accordance with formulae (3) and (10) are performed by the method of successive approximations. First, we posit in formula (3) approximately: $\sin A_{21}\approx\sin\alpha_{21}$ [or $\sin A_{12}\approx\sin\alpha_{12}$ in formula (10)], i.e., we assume, as a first approximation, that $(A_{21}-\alpha_{21})_I=0$. Then we find $(A_{21}-\alpha_{21})_{II}$:

$$(A_{21})_{II}=(A_{21}-\alpha_{21})_{II}+\alpha_{21} \tag{13}$$

and so on until, for practical purposes $(A_{21})_{n+1} = (A_{21})_n$. It is also possible to proceed in a different manner. Having found $(A_{21} - \alpha_{21})_{II}$, we may find the correction for $(A_{21})_{II}$ from the formula

$$\Delta (A_{21})_{II} = (A_{21} - \alpha_{21})''_{II} \cdot \sin (A_{21} - \alpha_{21}) \operatorname{ctg} \alpha_{21}, \tag{14}$$

which is accurate enough for distances up to 1000 km, and then to get

$$(A_{21})_{III} = (A_{21})_{II} + \Delta (A_{21})_{II}. \tag{15}$$

Formula (14) is easily obtained from (3). For distances exceeding 1000 km, the direct geodetic azimuth A_{12} should be calculated not from formula (7), but from formula (10). Formula (7) may be used in the process of successive approximations. Geodetic azimuths A_{12} and A_{21} may also be found by means the following group of formulae (cf. formulae (31) and (32) in [1] and formulae (3), (4), (10) and (11) of th

$$\left.\begin{aligned}
\operatorname{ctg} A_{12} &= \operatorname{ctg} \alpha_{12} - e^2\left(\sin B_2 - \frac{N_1}{N_2}\sin B_1\right)\frac{\cos B_1}{\cos B_2 \sin \Delta L_{12}}\\
\operatorname{ctg} \alpha_{12} &= \frac{\sin \Delta B_{12}}{\cos B_2 \sin \Delta L_{12}} + \sin B_1 \cdot \operatorname{tg}\frac{1}{2}\Delta L_{12}\\
\operatorname{ctg} A_{21} &= \operatorname{ctg} \alpha_{21} - e_2\left(\frac{N_2}{N_1}\sin B_2 - \sin B_1\right)\cdot\frac{\cos B_2}{\cos B_1 \sin \Delta L_{12}}\\
\operatorname{ctg} \alpha_{21} &= \frac{\sin \Delta B_{12}}{\cos B_1 \sin \Delta L_{21}} - \sin B_2 \operatorname{tg}\frac{1}{2}\Delta L_{12}
\end{aligned}\right\} \tag{I}$$

Relation (7) serves as a check on calculations of A_{12} and A_{21} from these formulae for distances not exceeding 1000 km

$$A_{12} - \alpha_{12} \approx A_{21} - \alpha_{21}.$$

For distances of over 1000 km, it is convenient to use the following control formulae:

$$\frac{\operatorname{ctg} A_{12} - \operatorname{ctg} \alpha_{12}}{\operatorname{ctg} A_{21} - \operatorname{ctg} \alpha_{21}} = \frac{N_1}{N_2} \cdot \frac{\cos^2 B_1}{\cos^2 B_2}.$$

These formulae are obtained on the basis of formulae I.

Relations (8) or (9) may likewise be used for verification.

Formulae I are simpler than those mentioned earlier. They do not require the use of the method of successive approximations, though the latter is not difficult, or the use of supplementary tables for the calculation of small quantities $(A - \alpha)$; the individual computational operations are simpler and fewer (for the larger distances). Furthermore, verification is significantly more simple. For this reason, this group of formulae is to be recommended for the calculation of geodetic azimuths A_{12} and A_{21}. This group is particularly suited for large distances (over 1000 km).

2. Formulae for the Solution of the Direct Geodetic Problem Technique of Calculations.

Problem: Given geodetic coordinates B_1 and L_1 of a point situated on the ellipsoid, the distance S_{12} from this point along a straight line to another point, also situated on the ellipsoid, and the geodetic azimuth A_{12} of a perpendicular section at the first point passing through the second point, it is required to find the geodetic coordinates B_2 and L_2 of the second point and the reverse geodetic azimuth A_{21} of the perpendicular section at the second point. Thus, B_1, L_1, S_{12}, A_{12} are known, and we are required to find B_2, L_2 and A_{21}.

After the substitution of relations (20) and (21) of paper [1] into formula (44) of the same paper, we obtain a formula for the calculation of the difference of the longitudes of the first and second points:

$$\operatorname{ctg} \Delta L_{12} = \frac{1}{\sin A_{12}} \left[\frac{N_1}{S_{12}} \cdot \frac{\cos B_1}{\sin z_{12}} + \cos B_1 \operatorname{ctg} z_{12} - \sin B_1 \cos A_{12} \right], \quad (16)$$

$$L_2 = L_1 + \Delta L_{12}. \quad (17)$$

Substituting expressions (21) and (23) of paper [1] into (45) of the same paper, we find the formula for calculating the difference in latitudes:

$$\sin \Delta B_{12} = \frac{S_{12}}{N_2} \left[\sin z_{12} \left(\cos A_{12} - \sin A_{12} \sin B_1 \operatorname{tg} \frac{1}{2} \Delta L_{12} \right) + e'^2 n_{12} \cos B_1 \right], \quad (18)$$

where $e'^2 = \frac{a^2 - b^2}{b^2}$.

The values of z_{12} and n_{12} in formulae (16) and (18) are found with the aid of formulae (22) and (33) in [1] by the method of successive approximations:

$$\cos z_{12} = -\frac{S_{12}}{2N_1}(1 + e'^2 n_{12}^2), \quad (19)$$

$$n_{12} = \cos z_{12} \sin B_1 + \sin z_{12} \cos B_1 \cos A_{12}. \quad (20)$$

The following sequence of operations is convenient in calculating differences of latitudes and longitudes. We may first approximate n_{12} by means of formula (20), by positing as a first approximation, in accordance with formula (19):

$$\cos z_{12} \approx -\frac{S_{12}}{2N_1}, \quad \sin z_{12} \approx 1.$$

With these data, we may determine $\cos z_{12}$ in the second approximation from formula (19), and then find $\sin z_{12}$ from the table of natural trigonometric functions. Then, if need be, we may determine n_{12} in the second approximation from formula (20), with $\cos z_{12}$ and $\sin z_{12}$ known to the second approximation. Then we find again $\cos z_{12}$, $\sin z_{12}$ and cotan z_{12}. For distances under 100 km, it is enough to get the first approximation in calculations of n_{12}, $\cos z_{12}$, $\sin z_{12}$ and cotan z_{12}.

The reverse geodetic azimuth A_{21} may be found by using formulae (3), (4) and (5) or the last two formulas of group I.

An effective check on calculations of B_2 and L_2 is provided by the formulae

$$S_{12} = \frac{N_2 \cos B_2 \sin \Delta L_{12}}{\sin A_{12} \sin z_{12}} \qquad (21)$$

pr

$$S_{12} = -\frac{N_1 \cos B_1 \sin \Delta L_{12}}{\sin A_{21} \sin z_{21}}. \qquad (22)$$

In the latter formula, $\sin z_{21}$ is determined from $\cos z_{21}$, which in turn is de[illegible] from the relation

$$\cos z_{21} = \frac{N_1}{N_2} \cos z_{12}. \qquad (23)$$

For checking the calculations of the reverse geodetic azimuth A_{21} and the quantities $\sin z_{12}$ and $\sin z_{21}$, we may use the exact formula:

$$N_1 \cos B_1 \sin A_{12} \sin z_{12} + N_2 \cos B_2 \sin A_{21} \sin z_{21} = 0, \qquad (24)$$

where $\sin z_{21}$ is determined by means of formula (23).

Formulae (21), (22), (23) and (24) have been obtained from formulae (15), (21), (38) and (39) in the paper already cited [1]. When azimuth A_{12} approaches zero, formula (1) should be used in the direct problem for purposes of verification.

The formulae for solving the direct and reverse geodetic problems have been reduced, insofar as possible, to a form convenient for computations. They are intended to be used with computing machines and tables of natural values of trigonometric quantities with eight significant figures. In the calculation of geodetic coordinates B and L, the formulae ensure a margin of error of the order of $\pm 0''.0001$, and of $\pm 0''.001$ for the calculation of geodetic azimuths. As a rule, calculations at all stages involve eight significant figures. In calculating certain supplementary terms, a fewer number of significant figures may be sufficient (see examples and calculation schedules).

Let us note that it is convenient to present each figure in the form of two factors, one of which is smaller than unity in absolute value, but greater than one tenth, while the second is 10 at a certain power. For example, the figures 123.45678 and 0.000 001 234 5786 are conveniently written as $0.12345678 \cdot 10^{+3}$ and $0.12345678 \cdot 10^{-5}$.

3. Practical Formulae for the Calculation of the Sides of a Triangle in an Astro-Geodetic Grid

In solving triangles, we assume that all their apexes are situated on the surface of the reference ellipsoid. Thus, we posit that the angles and sides measured on the physical surface of the Earth have already been transferred to the surface of the reference ellipsoid. The calculation of the sides of triangles by the Molodenskiy method may be performed in one of two ways. The first consists in the following. We calculate the reductions for the switch from spheroidal angles A_1, A_2 and A_3 (dihedral angles of perpendicular sections) to plane angles (of chords) α_1, α_2 and α_3 by means of formulae of sufficient accuracy (see formula (

$$\left.\begin{aligned} A_1-\alpha_1 &\approx \frac{\varepsilon''}{4}(1+\operatorname{ctg} A_2 \operatorname{ctg} A_3) \\ A_2-\alpha_2 &\approx \frac{\varepsilon''}{4}(1+\operatorname{ctg} A_1 \operatorname{ctg} A_3) \\ A_3-\alpha_3 &\approx \frac{\varepsilon''}{4}(1+\operatorname{ctg} A_1 \operatorname{ctg} A_2) \end{aligned}\right\}; \tag{25}$$

The spherical residue ε'' may be found from the well-known formula

$$\varepsilon'' = \rho'' \frac{S_{12}^2}{2R_m^2} \frac{\sin A_1 \sin A_2}{\sin A_3}, \tag{26}$$

where ρ'' is the number of seconds in a radian, and S_{12} is the known side of the triangle (chord).

The following relations serve as checks on the accuracy of the calculations of reductions by formula (25)

$$\operatorname{ctg} A_1 \operatorname{ctg} A_2 + \operatorname{ctg} A_1 \operatorname{ctg} A_3 + \operatorname{ctg} A_2 \operatorname{ctg} A_3 = 1,$$
$$(A_1-\alpha_1)+(A_2-\alpha_2)+(A_3-\alpha_3) \approx \varepsilon''. \tag{27}$$

Knowing the angles α_1, α_2, α_3 and one side S_{12} of a plane triangle, we may calculate the other sides by means of the theorem of sines:

$$S_{23} = S_{12}\frac{\sin\alpha_1}{\sin\alpha_3}, \quad S_{13} = S_{12}\frac{\sin\alpha_2}{\sin\alpha_3}. \tag{28}$$

The second method consists in the following. We reduce, in accordance with the indications given by Molodenskiy (cf. formula (74) in [1]) each of the three spheroidal angles by one fourth of the spherical residue. As Legendre's method, the two other sides may be found from the theorem of sines:

$$S_{23} = S_{12}\frac{\sin\left(A_1-\frac{\varepsilon}{4}\right)}{\sin\left(A_3-\frac{\varepsilon}{4}\right)}, \quad S_{13} = S_{12}\frac{\sin\left(A_2-\frac{\varepsilon}{4}\right)}{\sin\left(A_3-\frac{\varepsilon}{4}\right)}. \tag{29}$$

The accuracy of the second method is entirely adequate for triangles of any form with sides measuring up to 100 km. The second method allows the calculation of the quantities sought much more simply and rapidly, and it should therefore be used in practice.

Sequence of calculations of geodetic coordinates: we are given the geodetic coordinates B_1 and L_1 of one of the apexes of the triangle, the spheroidic angles A_1, A_2 and A_3, the base S_{12}, which is one of the sides of the triangle, and the direct geodetic azimuth A_{12}. We are requied to find the geodetic coordinates of the two other apexes. From the spheroidal angles A_2 and A_3 and side S_{12}, we calculate the spheric residue from formula (26). By solving the direct geodetic problem from the formulae given earlier, we find the geodetic coordinates B_2 and L_2 of the second apex of the triangle and the reverse geodetic azimuth A_{21} (see example and schedule of calculations). Then we calculate the two other sides S_{13} and S_{23} by means of formulae (29) and the direct geodetic azimuths A_{13} and A_{23} ($A_{13} = A_{12} - A_1$ and $A_{23} = A_{21} + A_2$). Finally, from the solutions of the two direct geodetic problems, we calculate twice the geodetic coordinates of the third apex of the triangle and the reverse azimuths A_{32} and A_{31}. In addition to the texts indicated for the direct geodetic problem, we may use as a check the agreement of the two determinations of the geodetic coordinates of the third point and the equation

$$A_{31} - A_{32} = A_3. \qquad (30)$$

4. Auxiliary tables

Tables for the calculation of natural trigonometric values of small angles with eight significant figures (from 0 to 6°)

Peters' eight-digit tables [2] give the natural values of trigonometric quantities for small angles (approx. to 6°) with a lesser number of significant figures. One small additional table allows the calculation of the tangent of a small angle (up to 2°30'). Finding the sought-for values of natural trigonometric quantities in the tables of small angles presents difficulties due to the inaccurate and laborious interpolation process. For this reason, it is advisable to compile tables which allow, through simple interpolation by means of a computing machine, to obtain natural trigonometric values to eight significant figures for sin x and tan x for small angles x, and to find, conversely, small angles x from sin x and tan x. The compilation of such tables is based on the following considerations. As we know, the relations $\frac{\sin x}{x}$, $\frac{\operatorname{tg} x}{x}$ и $\frac{x}{\sin x}$, $\frac{x}{\operatorname{tg} x}$

for small arcs x are close to unity and, when x is varied, vary much more gradually than x, sin x and tan x. By multiplying the values found in the tables for $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$ by x by means of a computer, we may find sin x and tan x, and by multiplying the tabular values for $\frac{x}{\sin x}$ and $\frac{x}{\tan x}$ by sin x and tan x, we find x. To compute tables for the relations $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$ to x = 6°, we may use developments of sin x and tan x into series by powers of x:

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots, \qquad (31)$$

$$\frac{\operatorname{tg} x}{x} = 1 + \frac{x^2}{3} + \frac{2x^4}{15} + \frac{17x^6}{315} + \cdots \qquad (32)$$

From tables of the relations $\frac{\sin x}{x}$ and $\frac{\tan x}{x}$, it is possible to compile tables of the reverse relations $\frac{x}{\sin x}$ and $\frac{x}{\tan x}$. We have prepared models of tables for the four relations mentioned. Values for the relations are given for every 10" within the interval from 0 to 1° (see Appendix 1).

Tables for the values of the major radii of curvature M and N of the Krasovskiy ellipsoid

The main radii of curvature M and N are required for the solution of the direct and reverse geodetic problems, as well as other fundamental geodetic problems. They are determined, as we know, from the exact formulae:

$$M = \frac{a(1-e^2)}{(1-e^2\sin^2 B)^{3/2}}, \tag{33}$$

$$N = \frac{a}{(1-e^2\sin^2 B)^{1/2}}. \tag{34}$$

Using the tables of A. A. Izotov and D. A. Larin [3], we have compiled a model for tables of the values of M and N by latitudes, every 1', from 56°50' to 57°. The table is presented in Appendix II.

The two tables proposed are sufficient for the solution of the direct and reverse problems for any given distance between points.

5. Examples and Schedules of Calculations

Example 1. Solution of reverse geodetic problem for distances from 0 to 1000 km ($S_{12} \approx 400$ km).

1	B_1	50°40'00",0000	6	$\frac{1}{2}\Delta B_{12}$	1 15 00 ,0000
2	B_2	53 10 00 ,0000	7	B_m	51 55 00 ,0000
3	L_1	70 00 00 ,0000	8	ΔL_{12}	4 00 00 ,0000
4	L_2	74 00 00 ,0000	9	$\frac{1}{2}\Delta L_{12}$	2 00 00 ,0000
5	ΔB_{12}	2 30 00 ,0000			

10	$\sin B_1$	0,773 47159	16	$\cos \frac{1}{2}\Delta B_{12}$	0,999 76203
11	$\cos B_1$	0,633 83097	17	$\sin B_m$	0,787 11448
12	$\sin B_2$	0,800 38274	18	$\sin \Delta L_{12}$	$0{,}697\ 56474 \cdot 10^{-1}$
13	$\cos B_2$	0,599 48934			
14	$\sin \Delta B_{12}$	$0{,}436\ 19387 \cdot 10^{-1}$	19	$\sin \frac{1}{2}\Delta L_{12}$	$0{,}348\ 99497 \cdot 10^{-1}$
15	$\sin \frac{1}{2}\Delta B_{12}$	$0{,}218\ 14885 \cdot 10^{-1}$	20	$\mathrm{tg} \frac{1}{2}\Delta L_{12}$	$0{,}349\ 20769 \cdot 10^{-1}$

calculation of distance S_{12}

21	N_1	$0{,}639\ 1053\ 9 \cdot 10^{+7}$
22	N_2	$0{,}639\ 1963\ 7 \cdot 10^{+7}$
23	$\frac{N_1}{N_2}$	1,000 14234
24	$\sin^2 \frac{\psi}{2}$	$0{,}938\ 68910 \cdot 10^{-3}$
25	$4 \frac{N_2}{N_1} \sin^2 \frac{\psi}{2}$	$0{,}375\ 52909 \cdot 10^{-2}$
26	$\frac{N_2}{N_1} \sin B_2 - \sin B_1$	$+\ 0{,}270\ 250 \cdot 10^{-1}$
27	$\left(\frac{N_2}{N_1} \sin B_2 - \sin B_1\right)^2$	$0{,}730\ 351 \cdot 10^{-3}$
28	$k_0 \left(\frac{N_2}{N_1} \sin B_2 - \sin B_1\right)^2$	$0{,}974\ 437 \cdot 10^{-5}$
29	$\frac{N_2}{N_1} - 1$	$+\ 0{,}1423 \cdot 10^{-3}$
30	$\left(\frac{N_2}{N_1} - 1\right)^2$	$0{,}2025 \cdot 10^{-7}$
31	$\left(\frac{S_{12}}{N_1}\right)^2$	$0{,}374\ 55667 \cdot 10^{-2}$
32	$\frac{S_{12}}{N_1}$	$0{,}612\ 01035 \cdot 10^{-1}$
33	S_{12}	$0{,}391\ 13911 \cdot 10^{+6}$

calculation α_{21}

34	$\cos B_1 \sin \Delta L_{12}$	$0{,}442\ 138\ 14 \cdot 10^{-1}$
35	$\frac{\sin \Delta B_{12}}{\cos B_1 \sin \Delta L_{12}}$	0,986 555 63
36	$\sin B_2 \,\mathrm{tg} \frac{1}{2}\Delta L_{12}$	$0{,}279\ 499\ 80 \cdot 10^{-1}$
37	$\mathrm{ctg}\, \alpha_{21}$	0,958 605 65
38	α_{21}	226°12′38″,675
39	$\sin \alpha_{21}$	$-\ 0{,}721\ 889\ 99$

calculation of distance $\alpha_{12} - \alpha_{21}$

40	$\mathrm{ctg} \frac{1}{2}(\alpha_{12} - \alpha_{21})$	$0{,}274\ 931\ 86 \cdot 10^{-1}$
41	$\alpha_{12} - \alpha_{21}$	176°51′01″,103

Calculation of Azimuths A_{21} and A_{12} (Variant I)

42	e^2	0,669 342·10−2
43	$\frac{N_2}{N_1}\sin B_2 - \sin B_1$	+0,270 250·10−1
44	$\frac{\cos B_2}{\cos B_1 \sin \Delta L_{12}}$	0,135 589·10+2
45	$\sin^2 \alpha_{21}$	0,521 125
46	$\sin (A_{21} - \alpha_{21})_I$	+0,127 814·10−3
47	$(A_{21} - \alpha_{21})_I$	+0°04′23″,636
48	ΔA_{21}	+0 ,323
49	$(A_{21} - \alpha_{21})_{II}$	+0°04′23″,959
50	$(A_{21})_{II}$	226°17′02″,634
51	A_{12}	43°08′03″,737

Check calculating of Azimuths A_{21} and A_{12}

$\sin A_{12}$	0,683 711 67
$\cos B_1$	0,633 830 97
$\sin A_{21}$	−0,722 774 97
$\cos B_2$	+0,599 489 34
Σ_I	+0,000 061 74
$\cos A_{12}$	+0,729 752 25
$\cos A_{21}$	−0,691 083 46
Σ_{II}	+0,0482 424
$\sin (A_{21} - \alpha_{21})$	+0,127 971·10−3
$\Sigma_{II} \sin (A_{21} - \alpha_{21})$	+0,000 061 74

Check of S_{12}, A_{12} and A_{21} *

n_0	0,43887	N_2	0,639 196 39·10+7
n_0^2	0,19261	$\cos B_2$	0,599 489 34
Δ_0	0,397 17·10−4	$\sin \Delta L_{12}$	+0,697 564 74·10−1
$(\cos z_{12})_0$	0,306 402 34·10−1	числитель	0,267 300 82·10+6
$(\sin z_{12})_0$	0,999 530 48	$\sin A_{12}$	+0,683 711 67
n_1	0,438623	$\sin z_{12}$	0,999 530 48
n_1^2	0,192 390	знаменатель	0,683 390 65
Δ_1	0,396 713·10−4	S_{12}	0,391 139 12·10+6
$(\cos z_{12})_I$	−0,306 401 88·10−1	д. б. S_{12}	0,391 139 11·10+6
$(\sin z_{12})_I$	0,999 530 48		

Calculation of geodetic Azimuths A_{12} and A_{21} from formulae of Group II (2nd Variant)

34	$\cos B_1 \sin \Delta L_{12}$	+0,442 138 14·10−1
35	$\frac{\sin \Delta B_{12}}{\cos B_1 \sin \Delta L_{12}}$	+0,986 555 63
36	$\sin B_2 \operatorname{tg} \frac{1}{2} \cdot \Delta L_{12}$	+0,279 499 80·10−1
37	$\operatorname{ctg} \alpha_{21}$	+0,958 605 65
38	α_{21}	226°12′38″,675
39	e^2	0,669 342·10−2
40	$\frac{N_2}{N_1} \sin B_2 - \sin B_1$	+0,270 250·10−1
41	$\frac{\cos B_2}{\cos B_1 \sin \Delta L_{12}}$	+0,135 589·10+2
42	$\operatorname{ctg} A_{21} - \operatorname{ctg} \alpha_{21}$	−0,245 267·10−2
43	$\operatorname{ctg} A_{21}$	+0,956 152 98
44	A_{21}	226°17′02″,636
45	$\cos B_2 \sin \Delta L_{12}$	+0,418 182 63·10−1
46	$\frac{\sin \Delta B_{12}}{\cos B_2 \sin \Delta L_{12}}$	+1,043 070 27
47	$\sin B_1 \operatorname{tg} \frac{1}{2} \Delta L_{12}$	+0,270 102 2·10−1

* For a distance $S_{12} < 100$ km it is enough to calculate $(\cos z_{12})_0$ and $(\sin z_{12})_0$

48	$\mathrm{ctg}\ \alpha_{12}$	+1,070 080 49
49	α_{12}	43°03′39″,778
50	e^2	0,669 342·10^{-2}
51	$\sin B_2 - \frac{N_1}{N_2}\sin B_1$	+0,270 212·10^{-1}
52	$\frac{\cos B_1}{\cos B_2 \sin \Delta L_{12}}$	+0,151 568·10^{+2}
53	$\mathrm{ctg}\ A_{12} - \mathrm{ctg}\ \alpha_{12}$	−0,274 132·10^{-2}
54	$\mathrm{ctg}\ A_{12}$	+1,067 339 17
55	A_{12}	43°08′03″,738
	Check: $A_{12} - \alpha_{12}$	0°04′23″,960
	$A_{21} - \alpha_{21}$	0 04 23 ,961

Example 2. Solution of direct geodetic problem for distances from 0 to 1000 km ($S_{12} \approx 400$ km).

1	B_1	50°40′0″,0000
2	L_1	70 00 0 ,0000
3	A_{12}	43°08′03″,737
4	S_{12}	0,391 139 11·10^{+6}
5	N_1	0,639 105 39·10^{+7}
6	$\frac{S_{12}}{N_1}$	0,612 010 34·10^{-1}
7	$\frac{S_{12}}{2N_1}$	0,306 005 17·10^{-1}
8	$\sin B_1$	0,773 471 59
9	$\cos B_1$	0,633 830 97
10	$\sin A_{12}$	+0,683 711 67
11	$\cos A_{12}$	+0,729 752 25
12	$\mathrm{tg}\ A_{12}$	+0,936 909 30
	e'^2	0,673 852 54 10^{-2}

Calculation of $\cos z_{12}$, $\sin z_{12}$, $\mathrm{ctg}\ z_{12}$

13	n_0	+0,438 87
14	n_0^2	0,192 61
15	Δ_0	0,397 17·10^{-4}
16	$(\cos z_{12})_0$	−0,306 402 34·10^{-1}
17	$(\sin z_{12})_0$	0,999 530 48
18	n_1	+0,438 623
19	n_1^2	0,192 390
20	Δ_1	0,396 713·10^{-4}
21	$(\cos z_{12})_I$	−0,306 401 88·10^{-1}
22	$(\sin z_{12})_I$	0,999 530 48
23	$\mathrm{ctg}\ z_{12}$	−0,306 545 78·10^{-1}

Calculation of L_2

24	$\frac{N_1}{S_{12}}$	0,163 395 93·10^{+2}
25	$\frac{N_1}{S_{12}}\frac{\cos B_1}{\sin z_{12}}$	0,103 614 05·10^{+2}
26	$\cos B_1\ \mathrm{ctg}\ z_{12}$	−0,194 298·10^{-1}
27	$\sin B_1 \cos A_{12}$	+0,564 442 6
28	$[Q_{12}]$	0,977 753 25·10^{+1}
29	$\mathrm{ctg}\ \Delta L_{12}$	+0,143 006 66·10^{+2}
30	ΔL_{12}	+4°00′00″,0000
31	L_2	74°00′00″,0000

Calculation of B_2

32	$\mathrm{tg}\frac{1}{2}\Delta L_{12}$	+0,349 207 69 10^{-1}
33	$\sin\frac{1}{2}\Delta L_{12}$	+0,348 994 97·10^{-1}
34	$\sin \Delta L_{12}$	+0,697 564 74 10^{-1}
35	$[\theta_{12}]$	0,711 285 05
36	$\sin z_{12}\,[\theta_{12}]$	0,710 951 09
37	$e'^2 n_1 \cos B_1$	0,1873397·10^{-2}
38	$[T_{12}]$	0,712 824 49
39	$\sin(\Delta B_{12})_0$	+0,436 255 96·10^{-1}
40	$(\Delta B_{12})_0$	+ 2°30′01″,2831
41	$(B_2)_0$	53°10′01″,2831
42	$(N_2)_0$	0,639 196 39·10^{+7}
43	$\left(\frac{N_1}{N_2}\right)_0$	0,999 857 63
44	$\sin(\Delta B_{12})_I$	+0,436 193 85·10^{-1}
45	$(\Delta B_{12})_I$	+ 2°29′59″,0998
46	N_2	0,639 196 37·10^{+7}

Note. The following designations are used below:

$$[Q_{12}] = \frac{N_1}{S_{12}}\frac{\cos B_1}{\sin z_{12}} + \cos B_1\ \mathrm{ctg}\ z_{12} - \sin B_1 \cos A_{12},$$

$$[\theta_{12}] = \cos A_{12} - \sin A_{12} \sin B_1\ \mathrm{tg}\frac{\Delta L_{12}}{2},$$

$$[T_{12}] = \sin z_{12}\,[\theta_{12}] + e' n_1 \cos B_1.$$

47	$\frac{N_1}{N_2}$	0,999 857 67
48	$\sin(\Delta B_{12})_{II}$	$+0,436\ 193\ 87\cdot10^{-1}$
49	(ΔB_{12})	+ 2°30′00″,0002
50	B_2	53°10′00″,0002

calculation of azimuth A_{21}

51	B_m	51°55′00″,0001
52	$\frac{1}{2}\Delta B_{12}$	+ 1 15 00 ,0001
53	$\sin B_m$	0,787 114 48
54	$\cos\frac{1}{2}\Delta B_{12}$	0,999 762 03
55	$\mathrm{tg}\frac{1}{2}\Delta L_{12}$	$+0,349\ 207\ 69\cdot10^{-1}$
56	$\mathrm{ctg}\frac{1}{2}(\alpha_{12}-\alpha_{21})$	$+0,274\ 931\ 86\cdot10^{-1}$
57	$\alpha_{12}-\alpha_{21}$	176°51′01″,103
58	A_{21}	226°17′02″,634

calculation of azimuth A_{21} (second variant)

51	$\sin B_2$	0,800 382 74
52	$\cos B_2$	0,599 489 34
53	$\cos B_1 \sin\Delta L_{12}$	$+0,442\ 138\ 14\cdot10^{-1}$
54	$\frac{\sin\Delta B_{12}}{\cos B_1 \sin\Delta L_{12}}$	+0,986 555 63
55	$\sin B_2\,\mathrm{tg}\frac{1}{2}\Delta L_{12}$	$+0,279\ 499\ 80\cdot10^{-1}$
56	$\mathrm{ctg}\,\alpha_{21}$	+0,958 605 65
57	α_{21}	226°12′38″,675
58	e^2	$0,669\ 342\cdot10^{-2}$
59	$\frac{N_2}{N_1}\sin B_2-\sin B_1$	$+0,270\ 250\cdot10^{-1}$
60	$\frac{\cos B_2}{\cos B_1 \sin\Delta L_{12}}$	$+0,135\ 589\cdot10^{+2}$
61	$\mathrm{ctg}\,A_{21}-\mathrm{ctg}\,\alpha_{21}$	$-0,245\ 267\cdot10^{-2}$
62	$\mathrm{ctg}\,A_{21}$	+0,956 152 98
63	A_{21}	226°17′02″,636

check calculations by formula (21)

N_2	$0,639\ 196\ 39\cdot10^{+7}$
$\cos B_2$	0,599 489 34
$\sin\Delta L_{12}$	$+0,697\ 564\ 74\cdot10^{-1}$
numerator ~~числитель~~	$0,267\ 300\ 82\cdot10^{+6}$
$\sin A_{12}$	+0,683 711 67
$\sin z_{12}$	0,999 530 48
Denominator ~~знаменатель~~	0,683 390 65
S_{12}	$0,391\ 139\ 12\cdot10^{+6}$
д. б. S_{12}	$0,391\ 139\ 11\cdot10^{+6}$

check calculations by formulae (23) and (24)

$\frac{N_1}{N_2}$	0,999 857 67
$\cos z_{12}$	$-0,306\ 401\ 88\cdot10^{-1}$
$\cos z_{21}$	$-0,306\ 358\ 27\cdot10^{-1}$
$\sin z_{21}$	0,999 530 62
N_1	$0,6391\ 053\ 9\cdot10^{+7}$
$\cos B_1$	0,633 830 97
$\sin A_{12}$	+0,683 711 67
$\sin z_{12}$	0,999 530 48
~~произведение~~ product I	+0,276 831 16
N_2	$0,639\ 196\ 37\cdot10^{+7}$
$\cos B_2$	0,5[illegible] 489 34
$\sin A_{21}$	−0,722 774 97
$\sin z_{21}$	0,999 530 62
~~произведение~~ product II	−0,276 831 16

Example 3. Solution of reverse Geodetic problem ($S_{12} \approx$ 7500 km) for distance exceeding 1000 km.

1	B_1	68°58′00″,0000	7	B_m	53°21′30″,0000
2	B_2	37 45 00 ,0000	8	ΔL_{12}	−155 31 00 ,0000
3	L_1	33 05 00 ,0000	9	$\frac{1}{2}\Delta L_{12}$	− 77 45 30 ,0000
4	L_2	−122 26 00 ,0000			
5	ΔB_{12}	− 31 13 00 ,0000	10	$\sin B_1$	0,933 371 78
6	$\frac{1}{2}\Delta B_{12}$	− 15 36 30 ,0000	11	$\cos B_1$	0,358 911 02
			12	$\sin B_2$	0,612 217 28

13	$\cos B_2$	0,790 689 57
14	$\sin \Delta B_{12}$	—0,518 275 80
15	$\sin \frac{1}{2} \Delta B_{12}$	—0,269 059 90
16	$\cos \frac{1}{2} \Delta B_{12}$	+0,963 123 44
17	$\sin B_m$	+0,802 383 68
18	$\sin \Delta L_{12}$	—0,414 428 53
19	$\sin \frac{1}{2} \Delta L_{12}$	—0,977 261 96
20	$\operatorname{tg} \frac{1}{2} \Delta L_{12}$	—0,460 895 36·10+1
21	N_1	0,639 692 31·10+7
22	N_2	0,638 626 08·10+7
23	$\frac{N_2}{N_1}$	0,998 333 22
24	$\sin^2 \frac{\psi}{2}$	0,343 421 62
25	$4 \frac{N_2}{N_1} \sin^2 \frac{\psi}{2}$	1,371 396 85
26	$\frac{N_2}{N_1} \sin B_2 - \sin B_1$	0,322 175
27	$\left(\frac{N_2}{N_1} \sin B_2 - \sin B_1\right)^2$	0,103 797
28	$k_0\left(\frac{N_2}{N_1} \sin B_2 - \sin B_1\right)^2$	0,138 486·10−2
29	$\frac{N_2}{N_1} - 1$	0,1667·10−3
30	$\left(\frac{N_2}{N_1} - 1\right)^2$	0,2778·10−5
31	$\left(\frac{S_{12}}{N_1}\right)^2$	1,370 014 77
32	$\frac{S_{12}}{N_1}$	1,170 476 30
33	S_{12}	0,748 744 69·10+7
34	$\cos B_1 \sin \Delta L_{12}$	—0,148 742 97
35	$\frac{\sin \Delta B_{12}}{\cos B_1 \sin \Delta L_{12}}$	+0,348 437 17·10+1
36	$\sin B_2 \operatorname{tg} \frac{1}{2} \Delta L_{12}$	—0,282 168 10·10+1
37	$\operatorname{ctg} \alpha_{21}$	+0,630 605 27·10+1
38	α_{21}	9°00′38″,907
39	$\sin \alpha_{21}$	+0,156 620 76

39-a

40	$\operatorname{ctg} \frac{1}{2}(\alpha_{12} - \alpha_{21})$	—0,383 974 58·10+1
41	$\alpha_{12} - \alpha_{21}$	330°48′17″,995
42	α_{12}	339 48 56 ,902

$\cos B_2 \sin \Delta L_{12}$	—0,327 684 32
$\frac{\sin \Delta B_{12}}{\cos B_2 \sin \Delta L_{12}}$	+1,581 631 37
$\sin B_1 \operatorname{tg} \frac{1}{2} \Delta L_{12}$	—0,430 186 72·10+1
$\operatorname{ctg} \alpha_{12}$	—2,720 235 86
α_{12}	339°48′56″,902
$\sin \alpha_{12}$	—0,345 039 29

Calcu

	e^2	0,669 342 16·10−2
43	$\frac{N_2}{N_1} \sin B_2 - \sin B_1$	—0,322 174 93
44	$\frac{\cos B_2}{\cos B_1 \sin \Delta L_{12}}$	—0,531 581 14·10+1
45	$\sin^2 \alpha_{21}$	0,245 300 62·10−1
46	$\sin (A_{21} - \alpha_{21})_0$	+0,281 195 34·10−3
47	$(A_{21} - \alpha_{21})_0$	+ 0°00′58″,001
48	ΔA_{21}	+ 0″,103
49	$(A_{21} - \alpha_{21})''_I$	+0°00′58″,104
50	$(A_{21})_I$	9°01′37″,011
51	$\sin (A_{21})_I$	0,156 898 98
52	$\frac{\sin (A_{21})_I}{\sin \alpha_{21}}$	1,001 776 39
53	$\sin (A_{21} - \alpha_{21})_{II}$	0,281 694 86·10−3
54	$(A_{21} - \alpha_{21})_{II}$	+0°00′58″,104
55	$(A_{21})_{II}$	9°01′37″,011

Calcul

56	$\frac{N_1}{N_2}$	1,001 669 57
57	e^2	0,669 342 16·10−2
58	$\sin B_2 - \frac{N_1}{N_2} \sin B_1$	—0,322 712 83
59	$\frac{\cos B_1}{\cos B_2 \sin \Delta L_{12}}$	—1,095 295 07
60	$\sin^2 \alpha_{12}$	0,119 052 11
61	$\sin (A_{12} - \alpha_{12})_0$	+0,281 664 85
62	$(A_{12} - \alpha_{12})_0$	+ 0°00′58″,098
63	ΔA_{12}	— 0″,045
64	$(A_{12} - \alpha_{12})''_I$	0°00′58″,053

Calculation of α_{21} (cont'd)

$(A_{12})_I$	339°49'54",955
$\sin (A_{12})_I$	−0,344 775 10
$\frac{\sin (A_{12})_I}{\sin \alpha_{12}}$	0,999 234 32

Calculation of azimuth A_{12} (cont'd)

68	$\sin (A_{12}-\alpha_{12})_{II}$	0,281 449 18·10−3
69	$(A_{12}-\alpha_{12})''_{II}$	+ 0°00'58",053
70	$(A_{12})_{II}$	339°49'54",955

Calculation of geodetic azimuths A_{12} and A_{21} by formulas of group II (2nd variant)

34	$\cos B_1 \sin \Delta L_{12}$	−0,148 742 97	45	$\cos B_2 \sin \Delta L_{12}$	−0,327 684 32
35	$\frac{\sin \Delta B_{12}}{\cos B_1 \sin \Delta L_{12}}$	+0,348 437 17·10+1	46	$\frac{\sin \Delta B_{12}}{\cos B_2 \sin \Delta L_{12}}$	+1,581 631 37
36	$\sin B_2 \operatorname{tg} \frac{1}{2} \Delta L_{12}$	−0,282 168 10·10+1	47	$\sin B_1 \operatorname{tg} \frac{1}{2} \Delta L_{12}$	−0,430 186 72·10+1
37	$\operatorname{ctg} \alpha_{21}$	+0,630 605 27·10+1	48	$\operatorname{ctg} \alpha_{12}$	−0,272 023 59·10+1
38	α_{21}	9°00'38",907	49	α_{12}	339°48'56",902
39	e^2	0,669 342 16·10−2	50	e^2	0,669 342 16·10−2
40	$\frac{N_2}{N_1} \sin B_2 - \sin B_1$	−0,322 174 93	51	$\sin B_2 - \frac{N_1}{N_2} \sin B_1$	−0,322 712 83
41	$\frac{\cos B_2}{\cos B_1 \sin \Delta L_{12}}$	−0,531 581 14·10+1	52	$\frac{\cos B_1}{\cos B_2 \sin \Delta L_{12}}$	−1,095 295 07
42	$\operatorname{ctg} A_{21} - \operatorname{ctg} \alpha_{21}$	−0,114 632 95·10−1	53	$\operatorname{ctg} A_{12} - \operatorname{ctg} \alpha_{12}$	−0,236 589 54·10−3
43	$\operatorname{ctg} A_{21}$	+0,629 458 94·10+1	54	$\operatorname{ctg} A_{12}$	−0,272 260 18·10+1
44	A_{21}	9°01'37",011	55	A_{12}	339°49'54",956

Check:

$\operatorname{ctg} A_{21} - \operatorname{ctg} \alpha_{21}$	−0,114 632 95·10−1	$\cos^2 B_2$	0,625 190 00
$\operatorname{ctg} A_{12} - \operatorname{ctg} \alpha_{12}$	−0,236 589 54·10−2	$\cos^2 B_1$	0,128 817 12
Proportion	+0,484 522 48·10+1	Proportion:	0,485 331 45·10+1
$\frac{N_2}{N_1}$	0,998 333 22	$\frac{N_2}{N_1} \cdot \frac{\cos^2 B_2}{\cos^2 B_1}$	+0,484 522 51·10+1

Example 4. Solution of direct geodetic problem for a distance exceeding 1000 km ($S_{12} \approx 7500$ km)

1	B_1	68°58'00",0000
2	L_1	33 05 00 ,0000
3	A_{12}	339 49 54 ,955
4	S_{12}	0,748 744 69·10+7
5	N_1	0,639 692 31·10+7
6	$\frac{S_{12}}{N_1}$	1,170 476 30
7	$\frac{S_{12}}{2N_1}$	0,585 238 15
8	$\sin B_1$	0,938 371 78
9	$\cos B_1$	0,358 911 02
10	$\sin A_{12}$	−0,344 775 10
11	$\cos A_{12}$	+0,938 685 31
12	$\operatorname{tg} A_{12}$	−0,367 295 73
	e'^2	0,673 852 54·10−2

Calculation of $\cos z_{12}$, $\sin z_{12}$ and cotan z_{12}

13	n_0	−0,209 340
14	n_0^2	0,438 233 10−1
15	Δ_0	0,172 824 10−3
16	$(\cos z_{12})_0$	−0,585 410 97
17	$(\sin z_{12})_0$	0,810 736 70
18	n_1	−0,273 265
19	n_1^2	0,746 739·10−1
20	Δ_1	0,294 487·10−3
21	$(\cos z_{12})_I$	−0,585 532 64
22	$(\sin z_{12})_I$	0,810 648 83
23	n^2	−0,273 408
24	n_{22}	0,747 521·10−1
25	Δ_2	0,294 796·10−3
26	$(\cos z_{12})_{II}$	−0,585 532 95

27	$(\sin z_{12})_{II}$	0,810 648 61
28	$\mathrm{ctg}\, z_{12}$	—0,722 301 81

Calculation of L_2

29	$\frac{N_1}{S_{12}}$	0,854 353 05
30	$\frac{N_1}{S_{12}} \frac{\cos B_1}{\sin z_{12}}$	0,378 260 96
31	$\cos B_1\, \mathrm{ctg}\, z_{12}$	—0,259 242 08
32	$\sin B_1 \cos A_{12}$	+0,876 142 38
33	$[Q_{12}]$	0,757 123 50
34	$\mathrm{ctg}\, \Delta L_{12}$	+0,219 599 24·10+1
35	ΔL_{12}	—155°31′00″,003
36	L_2	—122 26 00 ,003

Calculation of B_2

37	$\sin \Delta L_{12}$	—0,414 428 52
38	$\frac{1}{2} \Delta L_{12}$	—77°45′30″,002
39	$\mathrm{tg} \frac{1}{2} \Delta L_{12}$	—0,460 895 38·10+1
40	$[\Theta_{12}]$	—0,544 591 46
41	$\sin z_{12} [\Theta_{12}]$	—0,441 391 25
42	$e'^2 n \cos B_1$	—0,661 248 10—3
43	$[T_{12}]$	—0,442 052 50
44	$\sin (\Delta B_{12})_0$	—0,517 411 97
45	$(\Delta B_{12})_0$	—31°09′32″
46	$(B_2)_0$	37 48 28
47	$(N_2)_0$	0,638 628 19·10+7
48	$\left(\frac{N_1}{N_2}\right)_0$	1,001 666 3
49	$\sin (\Delta B_{12})_I$	—0,518 274 13
50	$(\Delta B_{12})_I$	—31°12′59″,60
51	$(B_2)_I$	37 45 00 ,40
52	$(N_2)_I$	0,638 626 08·10+7
53	$\left(\frac{N_1}{N_2}\right)_I$	1,001 669 57
54	$\sin (\Delta B_{12})_{II}$	—0,518 275 83
55	$(\Delta B_{12})_{II}$	—31°13′00″,006
56	$(B_2)_{II}$	37 44 59 ,994

Calculation of α_{21}, $\sin \alpha_{21}$

57	$\sin B_2$	0,612 217 26
58	$\cos B_2$	0,790 689 58
59	$\cos B_1 \sin \Delta L_{12}$	—0,148 742 96
60	$\frac{\sin \Delta B_{12}}{\cos B_1 \sin \Delta L_{12}}$	+0,348 437 22·10+1
61	$\sin B_2\, \mathrm{tg} \frac{1}{2} \Delta L_{12}$	—0,282 168 11·10+1
62	$\mathrm{ctg}\, \alpha_{21}$	+0,630 605 33·10+1
63	α_{21}	9°00′38″,905
64	$\sin \alpha_{21}$	+0,156 620 75

Calculation of azimuth A_{21}

	e^2	0,669 342 16·10—2
65	$\frac{N_2}{N_1}$	0,998 333 21
66	$\frac{N_2}{N_1} \sin B_2 - \sin B_1$	—0,322 174 96
67	$\frac{\cos B_2}{\cos B_1 \sin \Delta L_{12}}$	—0,531 581 17·10+1
68	$\sin^2 \alpha_{21}$	0,245 300 60·10—1
69	$\sin (A_{21} - \alpha_{21})_I$	+0,281 195 37·10—3
70	$(A_{21} - \alpha_{21})''_I$	+0°00′58″,001
71	ΔA_{21}	+ 0 ,103
72	$(A_{21} - \alpha_{21})_{II}$	+0 00 58 ,104
73	$(A_{21})_{II}$	9°01′37″,009
74	$\sin (A_{21})_{II}$	+0,156 898 97
75	$\frac{\sin (A_{21})_{II}}{\sin \alpha_{21}}$	1,001 776 39
76	$\sin (A_{21} - \alpha_{21})_{III}$	+0,281 694 88
77	$(A_{21} - \alpha_{21})_{III}$	+0°00′58″,104
78	A_{21}	9 01 37 ,009

Calculation of azimuth A_{21} (second variant)

57	$\sin B_2$	0,612 217 26
58	$\cos B_2$	0,790 689 58
59	$\cos B_1 \sin \Delta L_{12}$	—0,148 742 96
60	$\frac{\sin \Delta B_{12}}{\cos B_1 \sin \Delta L_{12}}$	+0,348 437 22·10+1
61	$\sin B_2 \cdot \mathrm{tg} \frac{1}{2} \Delta L_{12}$	—0,282 168 11·10+1
62	$\mathrm{ctg}\, \alpha_{21}$	+0,630 605 33·10+1
63	α_{21}	9°00′38″,905
64	e^2	0,669 342 16·10—2
65	$\frac{N_2}{N_1} \sin B_2 - \sin B_1$	—0,322 174 96
66	$\frac{\cos B_2}{\cos B_1 \sin \Delta L_{12}}$	—0,531 581 17·10+1
67	$\mathrm{ctg}\, A_{21} - \mathrm{ctg}\, \alpha_{21}$	—0,114 632 97·10—1
68	$\mathrm{ctg}\, A_{21}$	+0,629 459 00·10+1
69	A_{21}	9°01′37″,008

Check calculation by formula (21)

N_2	0,638 626 08·10+7
$\cos B_2$	0,790 689 57
$\sin \Delta L_{12}$	—0,414 428 53
Product I	—0,209 287 75·10+7
$\sin A_{12}$	—0,344 775 10
$\sin z_{12}$	0,810 648 61
Product II	—0,279 491 46
$I:II = S_{12}$	0,748 744 70·10+7
п. 6. S_{12}	0,748 744 69·10+2

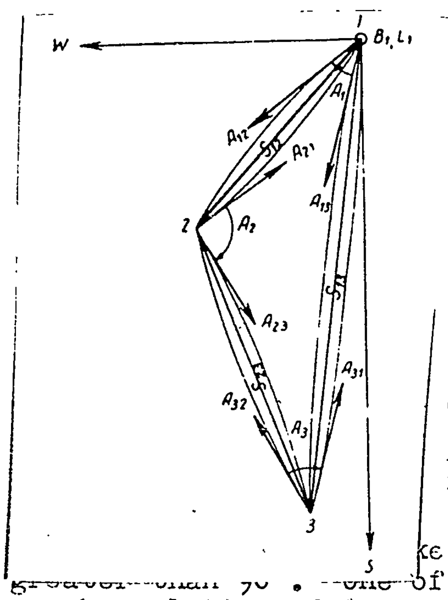
formulae (24) and (23)

$\frac{N_1}{N_2}$	1,001 669 57
$\cos z_{12}$	—0,585 532 95
$\cos z_{21}$	—0,586 510 48
$\sin z_{21}$	+0,809 941 85
N_1	0,639 621 31·10+7
$\cos B_1$	0,358 911 02
$\sin A_{12}$	—0,344 775 10
$\sin z_{12}$	0,810 648 61
Product I	—0,641 691 76
N_2	0,638 626 08·10+7
$\cos B_2$	0,790 689 58
$\sin A_{21}$	+0,156 898 97
$\sin z_{21}$	+0,809 941 85
Product II	+0,641 691 95
I + II	+0,000 000 19

Example 5. Calculation of geodetic coordinates of apexes and sides of triangle

In the triangle we know B_1, L_1, S_{12}, A_{12}, A_1, A_2 and A_3 and we seek S_{23}, S_{13}, A_{21}, A_{13}, A_{23}, A_{31}, A_{32}, B_2, L_2, B_3 and L_3 (see figure).

It is assumed that all the apexes of the triangle are situated on the surface of reference ellipsoid and that values S_{12}, A_1, A_2 and A_3 are reduced to zero elevation. We know that:



$$B_1 = 57°00'00'',0000$$
$$L_1 = 48\ 00\ 00\ ,0000$$
$$S_{12} = 39\,404,808\ м$$
$$A_{12} = 225°35'42'',281$$
$$A_1 = 36\ 35\ 40\ ,332$$
$$A_2 = 108\ 51\ 28\ ,110$$
$$A_3 = 34\ 32\ 55\ ,463$$

These initial data were obtained in the following manner. The geodetic coordinates of three points on the reference ellipsoid were selected. For a more effective check of the formulae, the sides of [illegible] ken as equal, and one of the angles is greater than 90°. One of the points is taken as a datum. Then, the solution of three reverse geodetic problems provides S_{12}, A_{12}, $A_1 = A_{12} - A_{13}$, $A_2 = A_{23} - A_{21}$, $A_3 = A_{31} - A_{32}$ and, as an additional check, the two other sides, S_{23} and S_{13}. These calculati ns are presented separatelt in Appendix III. The calculation of the geodetic coordinates of the apexes and sides of the triangle were performed by the formulae in paras 2 and 3 of the present paper. The sequence of calculations is indicated in para 3. The calculations are presented below.

1. Solution of Direct Geodetic Problem for Side S_{12}

Givens: B_1, L_1, S_{12}, A_{12} Sought: B_2, L_2, A_{21}

1	B_1	57°00'00".0000
2	L_1	48 00 00 ,0000
3	A_{12}	225°35'42",281
4	S_{12}	$0{,}394\ 048\ 08 \cdot 10^{+5}$
5	N_1	$0{,}639\ 331\ 23 \cdot 10^{+7}$
6	$\frac{S_{12}}{N_1}$	$0{,}616\ 344\ 17 \cdot 10^{-2}$
7	$\sin B_1$	0,838 670 57
8	$\cos B_1$	0,544 639 04
9	$\sin A_{12}$	— 0,714 412 57
10	$\cos A_{12}$	— 0,699 724 71
11	$\mathrm{tg}\, A_{12}$	$0{,}102\ 099\ 09 \cdot 10^{+1}$

Calculation of z_{12}

	e'^2	$0{,}673\ 852\ 54 \cdot 10^{-2}$
12	n_{12}	— 0,383 682
13	n^2_{12}	0,147 214
14	Δ_{12}	$0{,}305\ 708 \cdot 10^{-5}$
15	$\cos z_{12}$	$-\ 0{,}308\ 477\ 8 \cdot 10^{-3}$
16	$\sin z_{12}$	+ 0,99999525
17	$\mathrm{ctg}\, z_{12}$	$-\ 0{,}308\ 479 \cdot 10^{-3}$

Calculation of L_2

18	$\frac{N_1}{S_{12}}$	$0{,}162\ 247\ 01 \cdot 10^{+3}$
19	$\frac{N_1}{S_{12}} \frac{\cos B_1}{\sin z_{12}}$	$0{,}883\ 664\ 75 \cdot 10^{+2}$
20	$\cos B_1\, \mathrm{ctg}\, z_{12}$	$-\ 0{,}168\ 010 \cdot 10^{-3}$
21	$\sin B_1 \cos A_{12}$	— 0,586 838 5
22	$[Q_{12}]$	$0{,}889\ 516\ 33 \cdot 10^{+2}$
23	$\mathrm{ctg}\, \Delta L_{12}$	$-\ 1{,}245\ 101\ 74 \cdot 10^{+2}$
24	ΔL_{12}	— 0°27'36",5745
25	L_2	47°32'23",4255

Calculation of B_2

26	$\mathrm{tg}\, \frac{1}{2} \Delta L_{12}$	$-\ 0{,}401\ 567\ 12 \cdot 10^{-2}$
27	(θ_{12})	— 0,702 130 73
28	$\sin z_{12}\, (\theta_{12})$	— 0,702 127 39
29	$e'^2 n_{12} \cos B_1$	$-\ 0{,}140\ 813\ 7 \cdot 10^{-2}$
30	$[T_{12}]$	— 0,703 535 53
31	$\sin (\Delta B_{12})_0$	$-\ 0{,}433\ 620\ 03 \cdot 10^{-2}$
32	$(\Delta B_{12})_0$	— 0°14'54",4083
33	$(B_2)_0$	56°45'05",5917
34	$(N_2)_0$	$0{,}639\ 322\ 71 \cdot 10^{+7}$
35	$\left(\frac{N_1}{N_2}\right)_0$	1,00001333
36	$\sin (\Delta B_{12})_I$	$-\ 0{,}43362581 \cdot 10^{-2}$
37	$(\Delta B_{12})_I$	— 0°14'54",4202
38	B_2	56°45'05",5798

Calculation of A_{21}

39	$(B_m)_{12}$	56°52'32",7899
40	$\frac{1}{2} \Delta B_{12}$	— 0 07 27 ,2101
41	$\sin (B_m)_{12}$	0,837 487 74
42	$\cos \frac{1}{2} \Delta B_{12}$	+ 0,999 997 65
43	$\mathrm{tg}\, \frac{1}{2} \Delta L_{12}$	$-\ 0{,}401\ 567\ 12 \cdot 10^{-2}$
44	$\mathrm{ctg}\, \frac{1}{2} (\alpha_{12} - \alpha_{21})$	$-\ 0{,}336\ 308\ 33 \cdot 10^{-2}$
45	$\alpha_{12} - \alpha_{21}$	180°23'07",366
46	A_{21}	45°12'34",915

Check calculations by formula (21)

N_2	$0{,}639\ 322\ 71 \cdot 10^{+7}$
$\cos B_2$	0,548 270 61
$\sin \Delta L_{12}$	$-\ 0{,}803\ 121\ 28 \cdot 10^{-2}$
Numerator	$-\ 0{,}281\ 511\ 56 \cdot 10^{+5}$
$\sin A_{12}$	— 0,714 412 57
$\sin z_{12}$	+ 0,999 995 25
Denominator	— 0,714 409 18
S_{12}	$0{,}394\ 048\ 07 \cdot 10^{+5}$

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Check calculations by formulae (23) and (24)

$\frac{N_1}{N_2}$	1,0000 1333	N_1	0,639 331 23 · 10+7
$\cos z_{12}$	− 0,308 477 80 · 10−2	$\cos B_1$	0,544 639 04
$\cos z_{21}$	− 0,308 481 91 · 10−2	$\sin A_{12}$	−0,714 417 57
$\sin z_{21}$	+ 0,999 995 25	$\sin z_{12}$	0,999 995 25
		Product I	− 0,248 760 67 · 10+7
		N_2	0,639 322 71 · 10+7
		$\cos B_2$	0,548 270 61
		$\sin A_{21}$	+ 0,709 690 00
		$\sin z_{21}$	0,999 995 25
		Product II	− 0,248 760 67 · 10+7

2. Calculation of Direct Geodetic Azimuths A_{13} and A_{23} from Given A_{12}, A_{21} and A_1, A_2

A_{12}	225°35′42″,281	A_{21}	45°12′34″,917
A_1	36 35 40 ,332	A_2	108 51 28 ,110
A_{13}	189 00 01 ,949	A_{23}	154 04 03 ,027

3. Calculation of Spherical Residue of Triangle (1, 2, 3)

ρ	0,206 26·10+6
$\frac{1}{2}\left(\frac{S_{12}}{R_m}\right)^2$	0,190 34·10−4
$\sin A_1$	0,596 14
$\sin A_2$	0,946 32
$\sin A_3$	0,567 11
ε''	3″,9054
$\frac{\varepsilon''}{4}$	0″,9763

4. Calculation of Sides S_{23} and S_{13} by the First Method by Formulae (23), (25), (26) and (27)

Calculation of reduction for change-over from spheroidic angle to plane angle by formulae (25) and (27)

A_1 36°35′40″,332	ctg A_1 +1,346 769	ctg A_2 ctg A_3 −0,496 059	1+ctg A_2 ctg A_3 + 0,503 94
A_2 108 51 28 ,110	ctg A_2 −0,341 554	ctg A_1 ctg A_3 +1,955 993	1+ctg A_1 ctg A_3 + 2,955 99
A_3 34 32 55 ,463	ctg A_3 +1,452 360	ctg A_1 ctg A_2 −0,459 994	1+ctg A_1 ctg A_2 + 0,540 01
Σ = 180 00 03 ,905		Check: Σ = + 0,999 940	Σ = + 3,999 94

	Calculated	Corrections	Corrected
$A_1 - \alpha_1$	+ 0″,4919	+ 0″,0001	+ 0″,4920
$A_2 - \alpha_2$	+ 2 ,8856	+ 0 ,0004	+ 2 ,8860
$A_3 - \alpha_3$	+ 0 ,5272	+ 0 ,0002	+ 0 ,5274
Σ =	+ 3″,9047	+ 0″,0007	+ 3″,9054

Calculation of plane angles α_n

Measured angles		Reductions $A_n - \alpha_n$	Corrected angles α_n	
A_1	36°35′40″,332	− 0″,4920	α_1	36°35′39″,840
A_2	108 51 28 ,110	− 2 ,8860	α_2	108 51 25 ,224
A_3	34 32 55 ,463	− 0 ,5274	α_3	34 32 54 ,936
Σ	180 00 03 ,905	− 3″,9054		180 00 00 ,000

Calculation of sides of triangle S_{13} and S_{23} from given S_{12}, α_1, α_2, α_3.

S_{12}	$0,394\ 048\ 08 \cdot 10^{+5}$				
α_1	36°35′39″,840	$\sin\alpha_1$	0,596 146 41	S_{13}	$0,657\ 548\ 08 \cdot 10^{+5}$
α_2	108 51 25 ,224	$\sin\alpha_2$	0,946 328 15	S_{23}	$0,414\ 227\ 27 \cdot 10^{+5}$
α_3	34 32 54 ,936	$\sin\alpha_3$	0,567 104 98		

Check Calculation

$S_{12} : \sin\alpha_3$	$0,694\ 841\ 51 \cdot 10^{+5}$
$S_{13} : \sin\alpha_2$	$0,694\ 841\ 51 \cdot 10^{+5}$
$S_{23} : \sin\alpha_1$	$0,694\ 841\ 51 \cdot 10^{+5}$

5. Calculation of Sides S_{13} and S_{23} of Triangle by Second Method by Formulae (30) and (31)

Calculation of spheroidal angles modified through correction

$$A_1 - \frac{\varepsilon''}{4},\quad A_2 - \frac{\varepsilon''}{4},\quad A_3 - \frac{\varepsilon''}{4}$$

Measured angles		$\frac{\varepsilon''}{4}$	Corrected angles	
A_1	36°35′40″,332	− 0″,976	$A_1 - \frac{\varepsilon''}{4}$	36°35′39″,356
A_2	108 51 28 ,110	− 0 ,976	$A_2 - \frac{\varepsilon''}{4}$	108 51 27 ,134
A_3	34 32 55 ,463	− 0 ,976	$A_3 - \frac{\varepsilon''}{4}$	34 32 54 ,487
	Σ 180° 00′03″.905	− 2″, 928		180°00′00″,977

Calculation of sides S_{13} and S_{23} of triangle from given S_{12}, $A_1 - \frac{\varepsilon''}{4}$, $A_2 - \frac{\varepsilon''}{4}$, $A_3 - \frac{\varepsilon''}{4}$, by formulae (30) and (31)

S_{12}	$0,394\ 048\ 08 \cdot 10^{+5}$				
$A_1 - \frac{\varepsilon''}{4}$	36°35′39″,356	$\sin\left(A_1 - \frac{\varepsilon''}{4}\right)$	0,596 144 52	S_{23}	$0,414\ 227\ 27 \cdot 10^{+5}$
				S_{13}	$0,657\ 548\ 08 \cdot 10^{+5}$
$A_2 - \frac{\varepsilon''}{4}$	108 51 27 ,134	$\sin\left(A_2 - \frac{\varepsilon''}{4}\right)$	0,946 325 16		
$A_3 - \frac{\varepsilon''}{4}$	34 32 54 ,487	$\sin\left(A_3 - \frac{\varepsilon''}{4}\right)$	0,567 103 19		

Check calculation

$S_{12} : \sin\left(A_3 - \frac{\varepsilon}{4}\right)$	$0,694\ 843\ 70 \cdot 10^{+5}$
$S_{13} : \sin\left(A_2 - \frac{\varepsilon}{4}\right)$	$0,694\ 843\ 71 \cdot 10^{+5}$
$S_{23} : \sin\left(A_1 - \frac{\varepsilon}{4}\right)$	$0,694\ 843\ 71 \cdot 10^{+5}$

69. Solution of Direct Geodetic Problem for Side S_{23}

	Given	Sought
	B_2, L_2, S_{23}, A_{23}	B_3, L_3, A_{32}
1	B_2	56°45′05″,5798
2	L_2	47 32 23 ,4256
3	A_{23}	154 04 03 ,027
4	S_{23}	$0{,}414\ 227\ 27 \cdot 10^{+5}$
5	N_2	$0{,}639\ 322\ 71 \cdot 10^{+7}$
6	$\frac{S_{23}}{N_2}$	$0{,}64791577 \cdot 10^{-2}$
7	$\sin B_2$	0,836 300 99
8	$\cos B_2$	0,548 270 61
9	$\sin A_{23}$	+ 0,437 311 86
10	$\cos A_{23}$	— 0,899 309 93
11	$\mathrm{tg}\, A_{23}$	— 0,486 274 92

Calculation of z_{23}

	e'^2	$0{,}673\ 852\ 54 \cdot 10^{-2}$
12	n_{23}	— 0,495 774
13	n^2_{23}	0,245 792
14	Δ_{23}	$0{,}536\ 564 \cdot 10^{-5}$
15	$\cos z_{23}$	$-\ 0{,}324\ 494\ 45 \cdot 10^{-2}$
16	$\sin z_{23}$	+ 0,999 994 74
17	$\mathrm{ctg}\, z_{23}$	$-\ 0{,}324\ 496\ 16 \cdot 10^{-2}$

Calculation of L_3

18	$\frac{N_2}{S_{23}}$	$1{,}543\ 410\ 47 \cdot 10^{+2}$
19	$\frac{N_2}{S_{23}} \frac{\cos B_2}{\sin z_{23}}$	$+\ 0{,}846\ 211\ 05 \cdot 10^{+2}$
20	$\sin B_2 \cos A_{23}$	— 0,752 093 68
21	$\cos B_2\ \mathrm{ctg}\, z_{23}$	$-\ 0{,}177\ 911\ 71 \cdot 10^{-2}$
22	$[Q_{23}]$	$+\ 0{,}853\ 714\ 20 \cdot 10^{+2}$
23	$\mathrm{ctg}\, \Delta l_{23}$	$+\ 0{,}195\ 218\ 62 \cdot 10^{+3}$
24	Δl_{23}	+ 0°17′36″,5744
25	L_{23}	49°50′00″,0000

Calculation of B_3

26	$\mathrm{tg} \frac{1}{2} \Delta l_{23}$	$+\ 0{,}266\ 121\ 42 \cdot 10^{-2}$
27	$[\theta_{23}]$	— 0,900 246 63
28	$\sin z_{23} [\theta_{23}]$	— 0,900 241 89
29	$e'^2\, n_{23} \cos B_2$	$-\ 0{,}183\ 165 \cdot 10^{-2}$
30	$[T_{23}]$	— 0,902 073 54
31	$\sin (\Delta B_{23})_0$	$-\ 0{,}584\ 467\ 67 \cdot 10^{-2}$
32	$(\Delta B_{23})_0$	— 0°20′05″,5580
33	$(B_3)_0$	56°25′00″,0218
34	$(N_3)_0$	$0{,}639\ 311\ 15 \cdot 10^{+7}$
35	$\left(\frac{N_2}{N_3}\right)_0$	1,000 018 08
36	$\sin (\Delta B_{23})_I$	$-\ 0{,}584\ 478\ 24 \cdot 10^{-2}$
37	$(\Delta B_{23})_I$	— 0°20′05″,5798
38	B_3	56°25′00″,0000

Calculation of A_{32}

39	$(B_m)_{23}$	56°35′02″,7899
40	$\frac{1}{2} \Delta B_{23}$	— 0 10 02 ,7899
41	$(\sin B_m)_{23}$	0,834 695 15
42	$\cos \frac{1}{2} \Delta B_{23}$	+ 0,999 995 73
43	$\mathrm{tg} \frac{1}{2} \Delta L_{23}$	$+\ 0{,}256\ 121\ 42 \cdot 10^{-2}$
44	$\mathrm{ctg} \frac{1}{2} (\alpha_{23} - \alpha_{32})$	$+\ 0{,}213\ 784\ 22\ 10^{-2}$
45	$\alpha_{23} - \alpha_{32}$	179°45′18″,077
46	A_{32}	334°18′44″,950

Check calculations by formula (21)

N_3	$0{,}639\ 311\ 15\ 10^{+7}$
$\cos B_3$	0,553 149 24
$\sin \Delta L_{23}$	$+\ 0{,}512\ 239\ 48 \cdot 10^{-2}$
Product I	$+\ 0{,}181\ 145\ 54 \cdot 10^{+5}$
$\sin A_{23}$	+ 0,437 341 86
$\sin z_{23}$	0,999 994 74
Product II	+ 0,437,309 56
$I : II = S_{23}$	$0{,}414\ 227\ 26 \cdot 10^{+5}$

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Control calculations by formulae (23) and (24)

$\frac{N_2}{N_3}$	1,000 018 08
$\cos z_{23}$	$-$ 0,324 494 45 $\cdot 10^{-2}$
$\cos z_{32}$	$-$ 0,324 500 32 $\cdot 10^{-2}$
$\sin z_{32}$	0,999 994 74

N_2	0,639 322 71 $\cdot 10^{+7}$
$\cos B_2$	0,548 270 61
$\sin A_{23}$	+ 0,437 311 86
$\sin z_{23}$	0,999 994 74
Product I	+ 0,153 286 55
N_3	0,639 311 15 $\cdot 10^{+7}$
$\cos B_3$	0,553 149 24
$\sin A_{34}$	$-$ 0,433 462 71
$\sin z_{32}$	0,999 994 74
Product II	$-$ 0,153 286 55

7. Solution of Direct Geodetic Problem for Side S_{13}

	Given	Sought
	B_1, L_1, S_{13}, A_{13}	B_3, L_3, A_{31}
1	B_1	57°00′00″,0000
2	L_1	48 00 00 ,0000
3	A_{13}	189 00 01 ,949
4	S_{13}	0,657 548 08 $\cdot 10^{+5}$
5	N_1	0,639 331 23 $\cdot 10^{+7}$
6	$\frac{S_{13}}{N_1}$	1,028 493 60 $\cdot 10^{-2}$
7	$\sin B_1$	0,838 670 57
8	$\cos B_1$	0,544 639 04
9	$\sin A_{13}$	$-$ 0,156 443 80
10	$\cos A_{13}$	$-$ 0,987 686 86
11	$\operatorname{tg} A_{13}$	+ 0,158 394 13

Calculation of Z_{13}

	e'^2	0,673 852 54 $\cdot 10^{-2}$
12	η_{13}	$-$ 0,542 247
13	η^2_{13}	0,294 032
14	Δ_{13}	0,101 890 $\cdot 10^{-4}$
15	$\cos z_{13}$	$-$ 0,515 265 70 $\cdot 10^{-2}$
16	$\sin z_{13}$	+ 0,999 986 73
17	$\operatorname{ctg} z_{13}$	$-$ 0,516 292 54 $\cdot 10^{-2}$

Calculation of L_3

18	$\frac{N_1}{S_{13}}$	0,972 295 79 $\cdot 10^{+2}$
19	$\frac{N_1}{S_{13}} \frac{\cos B_1}{\sin z_{13}}$	+ 0,529 557 27 $\cdot 10^{+2}$
20	$\sin B_1 \cos A_{13}$	$-$ 0,828 343 90
21	$\cos B_1 \operatorname{ctg} z_{13}$	0,280 638 $\cdot 10^{-2}$
22	$[Q_{13}]$	+ 0,537 812 65 $\cdot 10^{+2}$
23	$\operatorname{ctg} \Delta L_{13}$	$-$ 0,343 773 71 $\cdot 10^{+3}$
24	ΔL_{13}	$-$ 0°10′00″,0000
25	L_3	47°50′00″,0000

Calculation B_3

26	$\operatorname{tg} \frac{1}{2} \Delta L_{13}$	$-$ 0,145 444 20 $\cdot 10^{-2}$
27	$[\Theta_{13}]$	$-$ 0,987 877 69
28	$\sin z_{13} [\Theta_{13}]$	$-$ 0,987 864 57
29	$e'^2 \eta_{13} \cos B_1$	$-$ 0,199 008 $\cdot 10^{-2}$
30	$[T_{13}]$	$-$ 0,989 854 65
31	$\sin (\Delta B_{13})_0$	$-$ 0,101 805 917 $\cdot 10^{-1}$
32	$(\Delta B_{13})_0$	$-$ 0°34′59″,9341
33	$(B_3)_0$	56°25′00″,0659
34	$(N_3)_0$	0,639 311 15 $\cdot 10^{+7}$
35	$\left(\frac{N_1}{N_3}\right)_0$	1,000 031 41
36	$\sin (\Delta B_{13})_1$	$-$ 0,101 809 114
37	$(\Delta B_{13})_1$	$-$ 0°35′00″,0000
38	B_3	56°25′00″,0000

Calculation of A_{31}

39	$(B_m)_{13}$	8°42′30″,0000
40	$\frac{1}{2} \Delta B_{13}$	$-$ 0 17 30 ,0000
41	$\sin (B_m)_{13}$	0,835 887 20
42	$\cos \frac{1}{2} \Delta B_{13}$	+ 0,999 987 04
43	$\operatorname{tg} \frac{1}{2} \Delta L_{13}$	$-$ 0,145 444 20 $\cdot 10^{-2}$
44	$\operatorname{ctg} \frac{1}{2} (\alpha_{13} - \alpha_{31})$	$-$ 0,121 576 52 $\cdot 10^{-3}$
45	$\alpha_{13} - \alpha_{31}$	$-$ 179°51′38″,461
46	A_{31}	8°51′40″,410

Check calculations by formula (21)

N_3	0,639 311 15 · 10^{+7}
$\cos B_3$	0,553 149 24
$\sin \Delta L_{13}$	− 0,290 887 79 · 10^{-2}
Product I	− 0,102 867 95 · 10^{+5}
$\sin A_{13}$	− 0,156 443 80
$\sin z_{13}$	0,999 986 73
Product II	− 0,156 441 72
$I : II = S_{13}$	0,657 548 06 · 10^{+5}

Check calculations by formulae (23) and (24

$\frac{N_1}{N_3}$	1,000 031 41	N_1	0,639 331 23 · 10^{+7}
$\cos z_{13}$	− 0,515 265 70 · 10^{-2}	$\cos B_1$	0,544 639 04
$\cos z_{31}$	− 0,515 281 88 · 10^{-2}	$\sin A_{13}$	− 0,156 443 80
$\sin z_{31}$	0,999 986 72	$\sin z_{13}$	+ 0,999 986 73
		Product I	− 0,544 737 51 · 10^{+6}
		N_3	0,639 311 15 · 10^{+7}
		$\cos B_3$	0,553 149 24
		$\sin A_{31}$	+ 0,154 041 74
		$\sin z_{31}$	0,999 986 72
		Product II	+ 0,544 737 48 · 10^{+6}

8. Calculation of Spheroidal Angle A_3 by Formula (32) (as check)

A_{31}	8°51′40″,410
A_{32}	334 18 44 ,950
A_3	34°32′55″,460
A_3	34°32′55″,463

Appendix 1

TABLES 1, 2, 3 and 4
for the Calculation of Natural Trigonometric Values for Small with Eight Significant Figures (Model from 0°59′ to 1°)

Explanations for Use of Tables 1, 2, 3 and 4

Tables 1 and 3 are intended for the calculation of sin x and tan x for a given value of x″, while Tables 2 and 4 are designed to provide values for x″ from given values for sin x and tan x. Tables 1 and 3 contain values for the proportions $\frac{\sin x}{x''}$ and $\frac{\tan x}{x''}$, while tables 2 and 4 provide values for the relations $\frac{x''}{\sin x}$ and $\frac{x''}{\tan x}$, where, for the sake of convenience, x is expressed in seconds rather than radians.

Example 1. It is required to find tan 0°59'17".2345. Here x" = 3557".2345. By interpolation, we find from Table 3 the relation for x" of $\frac{\tan x}{x''} \approx (0.484\ 861\ 55 + 20) \cdot 10^{-5} = 0.484\ 861\ 75\ 10^{-5}$. From a computer, we find the value sought:

Example 2. Find x", when tan x = $0.172\ 476\ 69 \cdot 10^{-1}$. On the basis of the rounded figure for $(\tan x)_0 = 0.172\ 48 \cdot 10^{-1}$ in Table 4 we find, by interpolation, the corresponding value

$$\frac{x''}{\tan x} = 206\ 244''.36.$$

On the computer we get:

$$x'' = (206\,244'',36 \times 0,172\,476\,69 \cdot 10^{-1}) = 3557'',2345.$$

Table 1. Values for $\frac{\sin x}{x''}$ (for the calculation of sin x from a given value for x")

x	$\frac{\sin x}{x''}$	x''
0°59'00"	0,4847 8988	3540"
10	974	50
20	961	60
30	948	70
40	934	80
50	920	90
1°00'00"	0,4847 8907	3600"

Table 2. Values for $\frac{x''}{\sin x}$ (for the calculation of x" from a given value for x)

x	$\frac{x''}{\sin x}$	$(\sin x)_0$
0°59'00"	206 274",93	0,017 [illegible]
10	4 ,99	210
20	5 ,05	259
30	5 ,10	307
40	5 ,16	355
50	5 ,22	404
1°00'00"	206 275",28	0,017 452

Table 3. Values for $\frac{\tan x}{x''}$ (for the calculation of tan x from a given value for x")

x	$\frac{\operatorname{tg} x}{x''}$	x''
0°59'00"	0,4848 6128	3540"
10	155	50
20	182	60
30	209	70
40	236	80
50	264	90
1°00'00"	0,4848 6291	3600"

Table 4. Values for $\frac{x''}{\tan x}$ (for the calculation of x" from a given value for tan x)

x	$\frac{x''}{\operatorname{tg} x}$	$(\operatorname{tg} x)_0$
0°59'00"	206 244",55	0,017 164
10	4 ,44	213
20	4 ,32	261
30	4 ,21	310
40	4 ,09	358
50	3 ,98	407
1°00'00"	206 243",86	0,017 455

Appendix 2

TABLE 5

Giving Values for the Main Radii of Curvature M and N of the Surface of the Krasovskiy Ellipsoid (Model from 56°50' to 57°)

Table 5

Values for the Main Radii of Curvature M and N of the Surface of the Krasovskiy Ellipsoid

B	M	d	N	d
56°50'	6 380 387,3	17,2	6 393 255,2	5,8
51	404,5	17,1	261,0	5,7
52	421,6	17,1	266.7	5,8
53	438,7	17,2	272,5	5,6
54	455,9	17,1	278,1	5,7
55	473,0	17,2	283,8	5,7
56	490,2	17,0	289,5	5,8
57	507,2	17,2	295,3	5,7
58	524,4	17,0	301,0	5,8
59	541,4	17,2	306,8	5,6
57°00'	6 380 558,6		6 393 312,4	

Appendix 3

CALCULATION OF INITIAL DATA OF EXAMPLE 5: S_{12}, A_{12}, A_1, A_2 and A_3 (AND FOR DIRECT VERIFICATION: S_{23}, S_{13} and α_1, α_2 and α_3)

The following values were taken for the coordinates of the apexes of the spheroidal

$$B_1 = 57°00'00'',0000 \qquad L_1 = 48°00'00'',0000$$
$$B_2 = 56°45'05'',5798 \qquad L_2 = 47°32'23'',4256$$
$$B_3 = 56°25'00'',0000 \qquad L_3 = 47°50'00'',0000$$

1. Solution of Reverse Geodetic Problem for Points 1 and 2

1	B_1	57°00'00'',0000
2	B_2	56 45 05 ,5798
3	L_1	48 00 00 ,0000
4	L_2	47 32 23 ,4256
5	ΔB_{12}	— 0 14 54 ,4202
6	$\frac{1}{2}\Delta B_{12}$	— 0 07 27 ,2101
7	$(B_m)_{12}$	56 52 32 ,7899
8	ΔL_{12}	— 0 27 36 ,5744
9	$\frac{1}{2}\Delta L_{12}$	— 0 13 48 ,2872
10	$\sin B_1$	0,838 670 57
11	$\cos B_1$	0,544 639 04
12	$\sin B_2$	0,836 300 99
13	$\cos B_2$	0,548 270 61
14	$\sin \Delta B_{12}$	—0,433 625 79·10⁻²
15	$\sin \frac{1}{2}\Delta B_{12}$	—0,216 813 41·10⁻²
16	$\cos \frac{1}{2}\Delta B_{12}$	0,999 997 65
17	$\sin (B_m)_{12}$	0,837 487 74
18	$\sin \Delta L_{12}$	—0,803 121 28·10⁻²
19	$\sin \frac{1}{2}\Delta L_{12}$	—0,401 563 88·10⁻²
20	$\mathrm{tg} \frac{1}{2}\Delta L_{12}$	0,401 567 12·10⁻²

Calculation of S_{12}

21	N_1	0,639 331 23·10⁺⁷
22	N_2	0,639 322 71·10⁺⁷
23	$\frac{N_2}{N_1}$	0,999 986 67

24	$\sin^2\frac{\psi}{2}$	0,951 599 10·10^{-5}
25	$4\frac{N_2}{N_1}\sin^2\frac{\psi}{2}$	0,380 634 57·10^{-4}
26	$\frac{N_2}{N_1}\sin B_2 - \sin B_1$	−0,238 073·10^{-2}
27	$\left(\frac{N_2}{N_1}\sin B_2 - \sin B_1\right)^2$	0,566 788·10^{-5}
28	$k_0\left(\frac{N_2}{N_1}\sin B_2 - \sin B_1\right)^2$	0,756 211·10^{-7}
29	$\frac{N_2}{N_1} - 1$	−0,1333·10^{-4}
30	$\left(\frac{N_2}{N_1} - 1\right)^2$	0,1777·10^{-9}
31	$\left(\frac{S_{12}}{N_1}\right)^2$	0,379 880 14·10^{-4}
32	$\frac{S_{12}}{N_1}$	0,616 344 17·10^{-2}
33	S_{12}	0,394 048 08·10^{+5}

Calculation of $\alpha_{12} - \alpha_{21}$

40	$\mathrm{ctg}\frac{1}{2}(\alpha_{12} - \alpha_{21})$	−0,336 308 33 10^{-2}
41	$\alpha_{12} - \alpha_{21}$	180°23′07″,336

Calculation of azimuths A_{12} and A_{21}

	e^2	+0,669 342·10^{-2}
42	$\frac{N_2}{N_1}\sin B_2 - \sin B_1$	−0,238 073·10^{-2}
43	$\frac{\cos B_2}{\cos B_1 \sin \Delta L_{12}}$	−0,125 344 10^{+3}
44	$\sin^2 \alpha_{21}$	0,502 655
45	$\sin(A_{21} - \alpha_{21})_I$	+0,100 400·10^{-2}
46	$(A_{21} - \alpha_{21})_I$	+ 0°03′27″,090
47	ΔA_{21}	+ 0″,207
48	$(A_{21} - \alpha_{21})_{II}$	+ 0°03′27″,297
49	A_{21}	45°12′34″,915
50	A_{12}	225°35′42″,281

Calculation of α_{21}

34	$\cos B_1 \sin \Delta L_{12}$	−0,437 411 20·10^{-2}
35	$\frac{\sin \Delta B_{12}}{\cos B_1 \sin \Delta L_{12}}$	+0,991 345 86
36	$\sin B_2\, \mathrm{tg}\frac{1}{2}\Delta L_{12}$	+0,003 358 31
37	$\mathrm{ctg}\, \alpha_{21}$	+0,994 704 19
38	α_{21}	45°09′07″,618
39	$\sin \alpha_{21}$	+0,708 981 60

Check calculations by formula (9)

$\sin A_{12}$	−0,714 412 57
$\cos A_{12}$	−0,699 724 71
$\cos B_1$	0,544 639 04
$\sin A_{21}$	+0,709 690 00
$\cos A_{21}$	+0,704 514 09
$\cos B_2$	0,548 270 61
$\sin A_{12} \cos B_1 + \sin A_{21} \cos B_2$	+0,000 005 19
$\sin(A_{21} - \alpha_{21})_{II}$	+0,100 500·10^{-2}
$D = \cos A_{12} \cos B_1 + \cos A_{21} \cos B_2$	0,000 516 70
$D \cdot \sin(A_{21} - \alpha_{21})_{II}$	+0,000 005 19

2. Solution of Reverse Geodetic Problem for Points 2 and 3

1	B_2	56°45′05″,5798	12	$\sin B_3$	0,833 082 18
2	B_3	56 25 00 ,0000	13	$\cos B_3$	0,553 149 24
3	L_2	47 32 23 ,4256	14	$\sin \Delta B_{23}$	−0,584 478 24·10^{-2}
4	L_3	47 50 00 ,0000	15	$\sin\frac{1}{2}\Delta B_{23}$	−0,292 240 37·10^{-2}
5	ΔB_{23}	− 0 20 05 ,5798			
6	$\frac{1}{2}\Delta B_{23}$	− 0 10 02 ,7899	16	$\cos\frac{1}{2}\Delta B_{23}$	+0,999 995 73
7	$(B_{23})_m$	56°35′02″,7899	17	$\sin(B_{23})_m$	0,834 695 15
8	ΔL_{23}	+ 0 17 36 ,5744	18	$\sin \Delta L_{23}$	+0,512 239 48·10^{-2}
9	$\frac{1}{2}\Delta L_{23}$	+ 0 08 48 ,2872	19	$\sin\frac{1}{2}\Delta L_{23}$	+0,256 120 58·10^{-2}
10	$\sin B_2$	0,836 300 99	20	$\mathrm{tg}\frac{1}{2}\Delta L_{23}$	+0,256 121 41·10^{-2}
11	$\cos B_2$	0,548 270 61			

Calculation of S_{23}

21	N_2	$0,639\ 322\ 71\cdot10^{+7}$
22	N_3	$0,639\ 311\ 15\cdot10^{+7}$
23	$\frac{N_3}{N_2}$	0,999 981 92
24	$\sin^2\frac{\psi}{2}$	$0,105\ 298\ 62\cdot10^{-4}$
25	$4\frac{N_3}{N_2}\sin^2\frac{\psi}{2}$	$0,421\ 166\ 86\cdot10^{-4}$
26	$\frac{N_3}{N_2}\sin B_3-\sin B_2$	$0,323\ 386\cdot10^{-3}$
27	$\left(\frac{N_3}{N_2}\sin B_3-\sin B_2\right)^2$	$0,104\ 578\cdot10^{-6}$
28	$k_0\left(\frac{N_3}{N_2}\sin B_3-\sin B_2\right)^2$	$0,139\ 528\cdot10^{-6}$
29	$\frac{N_3}{N_2}-1$	$0,1807\cdot10^{-4}$
30	$\left(\frac{N_3}{N_2}-1\right)^2$	$0,327\cdot10^{-9}$
31	$\left(\frac{S_{23}}{N_2}\right)^2$	$0,419\ 794\ 85\cdot10^{-4}$
32	$\frac{S_{23}}{N_2}$	$0,647\ 915\ 77\cdot10^{-2}$
33	S_{23}	$0,414\ 227\ 27\cdot10^{+5}$

Calculation of α_{32}

34	$\cos B_2\sin\Delta L_{23}$	$+0,280\ 845\ 85\cdot10^{-2}$
35	$\frac{\sin\Delta B_{23}}{\cos B_2\sin\Delta L_{23}}$	$-0,208\ 113\ 54\cdot10^{+1}$
36	$\sin B_3\,\mathrm{tg}\frac{1}{2}\Delta L_{23}$	$+0,213\ 370\ 19\cdot10^{-2}$
37	$\mathrm{ctg}\,\alpha_{32}$	$+0,208\ 326\ 91\cdot10^{+1}$
38	α_{32}	334°21′29″,898
39	$\sin\alpha_{32}$	−0,432 741 91

Calculation $\alpha_{23}-\alpha_{32}$

40	$\mathrm{ctg}\frac{1}{2}(\alpha_{23}-\alpha_{32})$	$+0,213\ 784\ 22\cdot10^{-2}$
41	$\alpha_{23}-\alpha_{32}$	179°45′18″,077

Calculation of azimuths A_{32} and A_{23}

	e^2	$0,669\ 342\cdot10^{-2}$
42	$\frac{N_3}{N_2}\sin B_3-\sin B_2$	$-0,323\ 386\cdot10^{-3}$
43	$\frac{\cos B_3}{\cos B_2\cdot\sin\Delta L_{23}}$	$0,196\ 958\cdot10^{+3}$
44	$\sin^2\alpha_{32}$	0,187 266
45	$\sin(A_{32}-\alpha_{32})_I$	$-0,798\ 365\cdot10^{-5}$
46	$(A_{32}-\alpha_{32})_I$	− 0°02′44″,675
47	ΔA_{32}	−0″,274
48	$(A_{32}-\alpha_{32})_{II}$	− 0°02′44″,949
49	A_{32}	334°18′44″,949
50	A_{23}	154 04 03 ,026

Check calculation by formula (9)

$\sin A_{23}$	+0,437 311 86
$\cos A_{23}$	−0,899 309 93
$\cos B_2$	0,548 270 61
$\sin A_{32}$	−0,433 462 71
$\cos A_{32}$	+0,901 171 50
$\cos B_3$	0,553 149 74
$\sin A_{23}\cos B_2+\sin A_{23}\cos B_3$	−0,000 004 33
$\sin(A_{32}-\alpha_{32})_{II}$	$-0,799\ 70\cdot10^{-5}$
$D=\cos A_{23}\cos B_2+\cos A_{32}\cos B_3$	$+0,541\ 71\cdot10^{-2}$
$D\sin(A_{32}-\alpha_{32})_{II}$	−0,000 004 33

3. Solution of Reverse Geodetic Problem for Points 3 and 1

1	B_3	56°25′00″,0000	6	$\frac{1}{2}\Delta B_{31}$	+ 0 17 30 ,0000
2	B_1	57 00 00 ,0000	7	$(B_{31})_m$	56 42 30 ,0000
3	L_3	47 50 00 ,0000	8	ΔL_{31}	+ 0 10 00 ,0000
4	L_1	48 00 00 ,0000	9	$\frac{1}{2}\Delta L_{31}$	0 05 00 ,0000
5	ΔB_{31}	+ 0 35 00 ,0000			

10	$\sin B_3$	0,833 082 18
11	$\cos B_3$	0,553 149 24
12	$\sin B_1$	0,838 670 57
13	$\cos B_1$	0,544 639 04
14	$\sin \Delta B_{31}$	$+0{,}101\ 809\ 11 \cdot 10^{-1}$
15	$\sin \frac{1}{2} \Delta B_{31}$	$+0{,}509\ 052\ 17 \cdot 10^{-2}$
16	$\cos \frac{1}{2} \Delta B_{31}$	$+0{,}999\ 987\ 04$
17	$\sin (B_{31})_m$	0,835 887 20
18	$\sin \Delta L_{31}$	$+0{,}290\ 887\ 79 \cdot 10^{-2}$
19	$\sin \frac{1}{2} \Delta L_{31}$	$+0{,}145\ 444\ 05 \cdot 10^{-2}$
20	$\mathrm{tg} \frac{1}{2} \Delta L_{31}$	$+0{,}145\ 444\ 20 \cdot 10^{-2}$

Calculation of S_{31}

21	N_1	$0{,}639\ 331\ 23 \cdot 10^{+7}$
22	N_3	$0{,}639\ 311\ 15 \cdot 10^{+7}$
23	$\frac{N_1}{N_3}$	1,000 031 41
24	$\sin^2 \frac{\psi}{2}$	$0{,}265\ 507\ 10 \cdot 10^{-4}$
25	$4 \frac{N_1}{N_3} \sin^2 \frac{\psi}{2}$	$1{,}062\ 061\ 76 \cdot 10^{-4}$
26	$\frac{N_1}{N_3} \sin B_1 - \sin B_3$	$+0{,}561\ 473 \cdot 10^{-2}$
27	$\left(\frac{N_1}{N_3} \sin B_1 - \sin B_3\right)^2$	$0{,}315\ 252 \cdot 10^{-4}$
28	$k_0 \left(\frac{N_1}{N_3} \sin B_1 - \sin B_3\right)^2$	$0{,}420\ 610 \cdot 10^{-6}$
29	$\frac{N_1}{N_3} - 1$	$0{,}3141 \cdot 10^{-4}$
30	$\left(\frac{N_1}{N_3} - 1\right)^2$	$0{,}987 \cdot 10^{-9}$
31	$\left(\frac{S_{31}}{N_3}\right)^2$	$1{,}057\ 865\ 53 \cdot 10^{-4}$
32	$\frac{S_{31}}{N_3}$	$1{,}028\ 525\ 90 \cdot 10^{-2}$
33	S_{31}	$0{,}657\ 548\ 08 \cdot 10^{+5}$

Calculation of α_{13}

34	$\cos B_3 \sin \Delta L_{31}$	$+0{,}160\ 904\ 36 \cdot 10^{-2}$
35	$\frac{\sin \Delta B_{31}}{\cos B_3 \sin \Delta L_{31}}$	$+0{,}632\ 730\ 60 \cdot 10^{+1}$
36	$\sin B_1 \,\mathrm{tg} \frac{1}{2} \Delta L_{31}$	$+0{,}121\ 979\ 77 \cdot 10^{-2}$
37	$\mathrm{ctg}\, \alpha_{13}$	$+0{,}632\ 608\ 22 \cdot 10^{+1}$
38	α_{13}	188°58′57″,856
39	$\sin \alpha_{13}$	$-0{,}156\ 136\ 89$

Calculation of $\alpha_{31} - \alpha_{13}$

40	$\mathrm{ctg} \frac{1}{2} (\alpha_{31} - \alpha_{13})$	$+0{,}121\ 576\ 57 \cdot 10^{-2}$
41	$\alpha_{31} - \alpha_{13}$	179°51′38″,461

Calculation of azimuths A_{13} and A_{31}

	e^2	$0{,}669\ 342 \cdot 10^{-2}$
42	$\frac{N_1}{N_3} \sin B_1 - \sin B_3$	$0{,}561\ 473 \cdot 10^{-2}$
43	$\frac{\cos B_1}{\cos B_3 \sin \Delta L_{31}}$	$0{,}338\ 486 \cdot 10^{+3}$
44	$\sin^2 \alpha_{13}$	$0{,}243\ 788 \cdot 10^{-1}$
45	$\sin (A_{13} - \alpha_{13})_I$	$+0{,}310\ 120 \cdot 10^{-3}$
46	$(A_{13} - \alpha_{13})_I$	+ 0°01′03″,967
47	$\Delta A_{13}''$	+ 0″,125
48	$(A_{13} - \alpha_{13})_{II}''$	+ 0°01′04″,092
49	A_{13}	189°00′01″,948
50	A_{31}	8°51′40″,409

Check calculation by formula (9)

$\sin A_{31}$	$+0{,}154\ 041\ 74$
$\cos A_{31}$	$+0{,}988\ 064\ 34$
$\cos B_3$	0,553 149 24
$\sin A_{13}$	$-0{,}156\ 443\ 80$
$\cos A_{13}$	$-0{,}987\ 686\ 86$
$\cos B_1$	0,544 639 04
$\sin A_{31} \cos B_3 + \sin A_{13} \cos B_1$	$+0{,}000\ 002\ 67$
$\sin (A_{13} - \alpha_{13})_{II}$	$+0{,}310\ 73 \cdot 10^{-3}$
$\cos A_{31} \cos B_3 + \cos A_{13} \cos B_1 = D$	$+0{,}861\ 42 \cdot 10^{-2}$
$D \sin (A_{13} - \alpha_{13})_{II}$	$+0{,}000\ 002\ 68$

4. Calculation of Spheroidal Angles A_1, A_2, A_3

№ точек		
1	A_{12}	225°35′42″,281
	A_{13}	189 00 01 ,948
	A_1	36 35 40 ,333
2	A_{23}	154°04′03″,028
	A_{21}	45 12 34 ,916
	A_2	108 51 28 ,112
3	A_{31}	8°51′40″,412
	A_{32}	334 18 44 ,949
	A_3	34 32 55 ,463

Uncorrected angles		Corrections	Corrected angles
A_1	36°35′40″,333	− 0″,001	36°35′40″,332
A_2	108 51 28 ,112	− 0 ,002	108 51 28 ,110
A_3	34 32 55 ,463	− 0 ,000	34 32 55 ,463
	180°00′03″,908	− 0″,003	180°00′03″,905

5. Calculation of Plane Angles of Triangle (1, 2, 3) (Directly from Known Sides S_{12}, S_{23}, S_{31})

Formulae:

$$p=\frac{1}{2}(S_{12}+S_{23}+S_{31});\quad Q=(p-S_{12})(p-S_{23})(p-S_{31}),$$

$$\operatorname{tg}\frac{\alpha_1}{2}=\frac{1}{p-S_{23}}\sqrt{\frac{Q}{p}},$$

$$\operatorname{tg}\frac{\alpha_2}{2}=\frac{1}{p-S_{31}}\sqrt{\frac{Q}{p}},$$

$$\operatorname{tg}\frac{\alpha_3}{2}=\frac{1}{p-S_{12}}\sqrt{\frac{Q}{p}}.$$

Check:

$$\alpha_1+\alpha_2+\alpha_3=180°,$$

$$\frac{S_{23}}{\sin\alpha_1}=\frac{S_{31}}{\sin\alpha_2}=\frac{S_{12}}{\sin\alpha_3}$$

S_{12}	39404,808	$p-S_{12}$	33886,364	Q	$81385619\cdot10^{+5}$
S_{13}	67754,808	$p-S_{31}$	$7536,363_5$	$\frac{Q}{p}$	11104423_2
S_{23}	41422,727	$p-S_{23}$	31868,444		
$2p$	146582,343	$\Sigma=p$	$73291,171_5$		
p	$73291,171_5$			$\sqrt{\frac{Q}{p}}$	10537753

$\operatorname{tg}\frac{\alpha_1}{2}$	0,33066418
$\operatorname{tg}\frac{\alpha_2}{2}$	1,3982543
$\operatorname{tg}\frac{\alpha_3}{2}$	0,31097325

		Correction
α_1	36°35′39″,844	″,842
α_2	108 51 25 ,224	224
α_3	34 32 54 ,934	934
$\Sigma\alpha_n$	180°00′00″,002	000

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A METHOD FOR THE SOLUTION OF THE REVERSE GEODETIC PROBLEM FOR GREAT DISTANCES THROUGH THE CALCULATION OF COORDINATES FOR THE "MEDIAN" POINT OF A GEODETIC LINE

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Formulae for the solution of the reverse geodetic problem are usually derived from a representation of an ellipsoid on a sphere, as proposed by Bessel. The length of a geodetic line is represented in the form of a series by powers of $e^2\sin^2\varphi$.

The form and complexity of the formula obtained depend, obviously, on the accuracy required. As we know, an increase in the number of terms in the series mentioned along with the increase of accuracy serves to complicate the formula considerably. This involves practical difficulties, since it is often necessary in practice to solve the reverse problem with intermediate accuracy.

We describe below a solution for the reverse geodetic problem which is of interest mainly for cases when it is necessary to determine not only the length of the geodetic line, but also the coordinates of its intermediate points, the length of the chord between its extreme points, the azimuth of a perpendicular section from one point to the other, and zenithal distances along the direction of the chord.

A calculation of this type may be performed with a predetermined margin of error. The solution is based on the following consideration. Since the flattening of the geodetic ellipsoid is not great, we may take as a first approximation the arc of a circle passing through the extreme points and deviating as little as possible from the geodetic line. To select a suitable circle to fit a geodetic line of considerable length, it is desirable to know the position of intermediate points. Such points may be found on the basis of a well-known property of the geodetic line which may be formulated in the following manner [2]. If we lower a perpendicular onto the surface of the ellipsoid from the middle of the chord connecting the extreme points of the geodetic line, the point of intersection of the perpendicular with the surface of the ellipsoid will lie on the geodetic line, close to its midpoint. Let us henceforth refer to that point as the "median" point*. The coordinates of this point are easily found from those of the end-points of the geodetic line. Knowing the position of the "median"

* Let us note that this point is, at the same time, the "median" point of alignment, i.e. the direct azimuths of the perpendicular sections at that point to the extreme points of the geodetic line will differ by exactly 180° (cf. Appendix, Example 2).

point and of the two end-points of the geodetic line, it is possible to determine the radius of the circle passing through these three points.

We will take the length of the arc of the circle thus plotted as the length of the geodetic line between the end-points.

In substituting the arc of a circle for a geodetic line of considerable length, the resulting error may be outside the permissible margin. In that case, the geodetic line should be divided into two segments, and the coordinates of the "median" points of each segment should be determined separately. Then a circle should be found in the manner described above for each portion of the geodetic line. The process may be repeated many times. It is thus possible to attain any given margin of error.

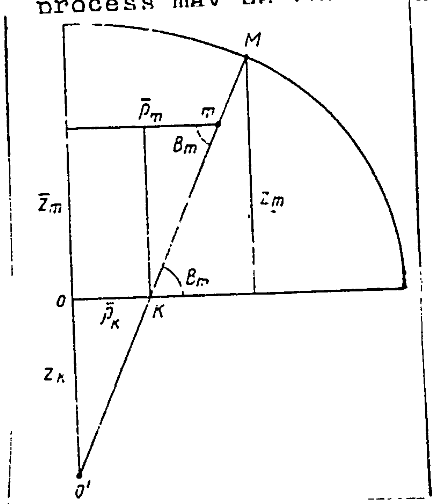

Fig. 1

Let us note that the need does not arise for the division of the geodetic line into many parts, since the error decreases rapidly as this is done. For example, it can be shown that if we use one "median" point for a geodetic line of a length of 4000 km, under the most unfavorable circumstances (see Appendix, Example 3) this length will be determined with a relative accuracy close to 1: 1,000,000. In determining the length of a line measuring 8000 km, three "median" points must be used to attain a comparable accuracy. The coordinates of the "median" point of a geodetic line are found in the following manner.

Since a perpendicular lowered onto the surface of the ellipsoid from the mid-point of the chord lies in the plane of the meridian of the "median" point of the geodetic line, the longitude of the "median" point will be defined as

$$\operatorname{tg} L_m = \frac{\bar{y}_m}{\bar{x}_m}, \qquad (1)$$

where $\bar{x}_m$, $\bar{y}_m$ and $\bar{z}_m$ are the coordinates of the mid-point of the chord which, in turn, are found from the coordinates of the end-points of the chord:

$$\bar{x}_m = \frac{x_1 + x_2}{2}; \quad \bar{y}_m = \frac{y_1 + y_2}{2}; \quad \bar{z}_m = \frac{z_1 + z_2}{2}. \qquad (2)$$

The origin of the system of rectangular coordinates lies at the center of the reference ellipsoid, the z axis coinciding with its short axis and the x axis being situated in the plane of the initial meridian.

Somewhat harder to find is the latitude u_m of the "median" point, needed for the solution.

From Fig. 1, which is in the plane of the meridian of the "median" point, we find

$$\operatorname{tg} B_m = \frac{\bar{z}_m + \bar{z}_k}{\bar{\rho}_m};$$

But

$$\bar{z}_k = z_m \left(1 - \frac{a^2}{b^2}\right) = -z_m \cdot e^2 \cdot \frac{a^2}{b^2},$$

and, therefore

$$\operatorname{tg} B_m = \frac{\bar{z}_m}{\bar{\rho}} + e^2 \cdot \frac{a^2}{b^2} \cdot \frac{z_m}{\bar{\rho}}.$$

Keeping in mind that

$$z_m = b \sin u_m; \quad \operatorname{tg} u_m = \sqrt{1-e^2} \cdot \operatorname{tg} B_m \quad \text{и} \quad \frac{a^2}{b^2} = \frac{1}{1-e^2},$$

we find

$$\operatorname{tg} u_m = \sqrt{1-e^2} \cdot \frac{\bar{z}_m}{\bar{\rho}_m} + e^2 \cdot \frac{a}{\bar{\rho}_m} \sin u_m. \qquad (3)$$

The derived latitude u_m is then found by the method of successive approximations. Since the second term on the right-hand side is considerably smaller than the first, the calculations are simple.

From the same figure, we may obtain a formula of somewhat different form which, in some cases, may be more convenient to use than the first:

$$\operatorname{ctg} u_m = \frac{1}{\sqrt{1-e^2}} \frac{\sqrt{\bar{x}_m^2 + \bar{y}_m^2}}{\bar{z}_m} - \frac{ae^2}{\sqrt{1-e^2}} \cdot \frac{1}{\bar{z}_m} \cos u_m, \qquad (4)$$

where

$$\bar{\rho}_m = \sqrt{\bar{x}_m^2 + \bar{y}_m^2}.$$

Knowing tan u_m and tan L_m, we find:

$$x_m = a \cos u_m \cos L_m,$$
$$y_m = a \cos u_m \sin L_m,$$
$$z_m = b \sin u_m.$$

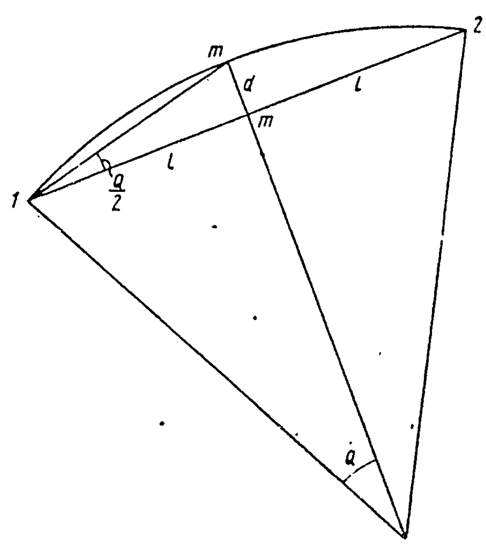

Fig. 2.

For the calculation of the length of the geodetic line, we obtain the formula (Fig. 2)

$$S_0 = 2l \cdot \frac{Q}{\sin Q}, \qquad (5)$$

where

$$\operatorname{tg}\frac{Q}{2} = \frac{d}{l}^{*}, \quad d = \sqrt{(\bar{x}_m - x_m)^2 + (\bar{y}_m - y_m)^2 + (\bar{z}_m - z_m)^2},$$

$$2l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

The formulae for the calculation of the azimuth of the perpendicular section have the following form (see similar formula (30) in [1], where longitude L at the initial point is taken to equals zero):

$$\operatorname{ctg}\bar{A}_{12} = \frac{(z_2 - z_1)\cos B_1 - [(x_2 - x_1)\cos L_1 + (y_2 - y_1)\sin L_1]\sin B_1}{(y_2 - y_1)\cos L_1 - (x_2 - x_1)\sin L_1} \qquad (6)$$

For the reverse azimuth $\bar{A}_{21}$, subscripts 1 and 2 in this formula should be transposed.

For $L_1 = 0$, these formulae become simpler.

The formulae for the determination of zenithal distances have the following form (see formulae (28) and (29) in [1]):

$$\operatorname{ctg}\bar{z}_{12} = \frac{[(x_2 - x_1)\cos L_1 + (y_2 - y_1)\sin L_1]\cos B_1 + (z_2 - z_1)\sin B_1}{(z_2 - z_1)\cos B_1 - [(x_2 - x_1)\cos L_1 + (y_2 - y_1)\sin L_1]\sin B_1}\cos\bar{A}_{12} \qquad (7)$$

or

$$\operatorname{ctg}\bar{z}_{12} = \frac{[(x_2 - x_1)\cos L_1 + (y_2 - y_1)\sin L_1]\cos B_1 + (z_2 - z_1)\sin B_1}{(y_2 - y_1)\cos L_1 - (x_2 - x_1)\sin L_1}\sin\bar{A}_{12}. \qquad (7a)$$

For ... 2 must be transposed in these formulae.

If the need arises to calculate the azimuth of the geodetic line, it may be found by known methods [3], using the value for S_0 found by means of formula (5) of the present paper.

The azimuth of the geodetic line may also be obtained by repeating, at one end-point of the geodetic line, the process of multiple subdivision. Then the azimuth of the perpendicular section of the smallest link will approach the azimuth of the geodetic line. It is then possible, by means of a known formula [2] to make the change-over to the azimuth of the geodetic line.

Knowing the azimuth of the geodetic line at one point only, we may find the azimuths for other points, by making use of the known relation:

$$c = \cos u_1 \sin A_1^s = \cos u_2 \sin A_2^s = \ldots = \cos u_m \sin A_m^s.$$

Below, we give examples of calculations by means of

---

*For calculating $y = \frac{Q}{\sin Q}$, it is convenient to compile a table for the variable $x = \frac{d}{l} = \tan\frac{Q}{2}$. Then $y = (1 \; x^2) \cdot \frac{\operatorname{arc\,tan} x}{x}$. If we assume that x varies within the limits 0.05 - 0.4, by developing in a series by powers of x, we have:

$$y = 1 + 2\left(\frac{x^2}{1\cdot 3} - \frac{x^4}{3\cdot 5} + \frac{x^6}{5\cdot 7} - \frac{x^8}{7\cdot 9} + \frac{x^{10}}{9\cdot 11} - \frac{x^{12}}{11\cdot 13} + \ldots\right).$$

of the formulae proposed. For S_o = 8000 km and one "median" point, the deviation of the result from the exact value of S_o equals ~100 m, and is reduced to ~6 m for two "median" points.

For a given length of the geodetic line, the greatest deviation from the exact value of S_o will occur when the geodetic line crosses areas of slight curvature variation, i.e. equatorial areas along a meridian. This case is illustrated by Example 3.

Example 1. S_o = 8000 km, one "median" point *.

1	B_1	68°58′00″,000
2	L_1	+ 33 05 00 ,000
3	B_2	37 45 00 ,000
4	L_2	—122 26 00 ,000

Calculation of S_o

5	$\operatorname{tg} B_1$	2,6005659
6	$\operatorname{tg} B_2$	0,77428273
7	$\sqrt{1-e^2}$	0,99664767
8	$\operatorname{tg} u_1$	2,5918480
9	$\operatorname{tg} u_2$	0,77168708
10	$\cos u_1$	0,35996206
11	$\cos u_2$	0,79168327
12	$\sin u_1$	0,93296694
13	$\sin u_2$	0,61093175
14	$\cos L_1$	+0,83787754
15	$\cos L_2$	—0,53631791
16	$\sin L_1$	+0,54585826
17	$\sin L_2$	—0,84401605
18	a	6378245,0
19	x_1	+1923705,0
20	y_1	+1253250,3
21	z_1	+5930743,0
22	b	6356863,0
23	x_2	—2708164,0
24	y_2	—4261901,2
25	z_2	+3883609,4
26	$\overline{x}_m$	—392229,5
27	$\overline{y}_m$	—1504325,4
28	$\overline{z}_m$	+4907176,2
29	$\operatorname{tg} L_m$	+0,38353194·10+1
30	$\cos L_m$	—0,25229951
31	$\sin L_m$	—0,96764919
32	$\sqrt{\overline{x}_m^2+\overline{y}_m^2}$	1554618,6
33	$\dfrac{\overline{z}_m}{\sqrt{\overline{x}_m^2+\overline{y}_m^2}}$	+0,31565145·10+1
34	$\sqrt{1-e^2}\cdot\dfrac{\overline{z}_m}{\sqrt{\overline{x}_m^2+\overline{y}_m^2}}$	+0,31459328₅ ·10+1
35	$\dfrac{ae^2}{\sqrt{\overline{x}_m^2+\overline{y}_m^2}}$	+0,02746157

36	$(\operatorname{tg} u_m)_0$	+0,31459328·10+1
37	$(\sin u_m)_0$	+0,95301140
38	$(\operatorname{tg} u_m)_I$	0,31721040·10+1
39	$(\sin u_m)_I$	0,95373226
40	$(\operatorname{tg} u_m)_{II}$	0,31721238·10+1
41	$(\sin u_m)_{II}$	0,95373133
42	$(\operatorname{tg} u_m)_{III}$	0,31721238·10+1
43	$\cos u_m$	0,30066018
44	$\sin u_m$	0,95373133
45	x_m	—483830,8
46	y_m	—1855645,7
47	z_m	+6062739,4
48	d^2	146714306
49	d	1211256,8
50	$(2l)^2$	560618625
51	$2l$	7487447,0
52	l	3743723,5
53	$\dfrac{d}{l}=\operatorname{tg}\dfrac{Q}{2}$	0,323543 34
54	$\dfrac{Q}{2}$	17°55′43″,112
55	$Q°$	35°51′26″,224
56	Q	0,62582766
57	$\sin Q$	0,58576829
58	$\dfrac{Q}{\sin Q}$	1,06838774
59	S_0	7999496,6

val S_o = 7999605,5 м

ΔS_0 — 108,9 м

lation of azimuths A_{12} and
f perpendicular section

60	$x_2 - x_1$	—4631869,0
61	$y_2 - y_1$	—5515151,5
62	$z_2 - z_1$	—2047133,6
63	$\cos B_1$	+0,35891102
64	$\sin B_1$	+0,93337178
65	$\cos L_1$	+0,83787754
66	$\sin L_1$	+0,54585826

* Example taken from [3].

67	$(x_2-x_1)\cos L_1+(y_2-y_1)\sin L_1$	−6891430,0
68	Numerator	+5697527,5
69	Denominator	−2092677,6
70	$\mathrm{ctg}\,\bar{A}_{12}$	−2,7226017
71	$\bar{A}_{12}$	339°49′54″,953
	Exact val.	339°49′54″,956
72	$\cos B_2$	0,79068957
73	$\sin B_2$	0,61221728
74	$\cos L_2$	−0,53631791
75	$\sin L_2$	−0,84401605
76	$(x_1-x_2)\cos L_2+(y_1-y_2)\sin L_2$	+7139030,7
77	Numerator	−5989285,1
78	Denominator	−951497,3
79	$\mathrm{ctg}\,\bar{A}_{21}$	+6,2945897
80	$\bar{A}_{21}$	9°01′37″,010
	Exact $\bar{A}_{21}$	9°01′37″,011

Calculation of zenithal distance z_{12}

81	$(x_2-x_1)\cos L_1+(y_2-y_1)\sin L_1$	−6891430,0
82	Numerator	−4384146,9
83	Denominator	+5697527,5
84	Fraction	−0,76948236
85	$\cos \bar{A}_{12}$	+0,93868531
86	$\mathrm{ctg}\, z_{12}$	−0,72230179
87	z_{12}	144°09′33″,664
	Exact z_{12}	144°09′33″,661

Example 2. Geodetic line S_0 = 8000 km, divided into two portions S_I and S_{II}

		For portion S_I	For portion S_{II}
1	x_1	+1923705,0	−483830,8
2	y_1	+1253250,3	−1855645,7
3	z_1	+5930743,0	+6062739,4
4	x_2	−483830,8	−2708164,0
5	y_2	−1855645,7	−4261901,2
6	z_2	+6062739,4	+3883609,4
7	$\bar{x}_m$	+719937,2	−1595997,4
8	$\bar{y}_m$	−301197,7	−3058773,4
9	$\bar{z}_m$	+5966741,2	+4973174,4
10	$\mathrm{tg}\, L_m$	−0,41836663	+1,91652781
11	$\cos L_m$	+0,92251913	−0,46259235
12	$\sin L_m$	−0,38585124	−0,88657112
13	$\frac{\bar{x}_m}{\cos L_m}=\sqrt{\bar{x}_m^2+\bar{y}_m^2}$	+780403,5	+3450116,2

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14	$\frac{\bar{z}_m}{\sqrt{\bar{x}_m^2+\bar{y}_m^2}}$	+7,6841547	+1,44145128
15	$\sqrt{1-e^2}\cdot\frac{\bar{z}_m}{\sqrt{\bar{x}_m^2+\bar{y}_m^2}}$	+7,6583949	+1,43661906
16	$\frac{ae^2}{\sqrt{\bar{x}_m^2+\bar{y}_m^2}}$	+0,05470538	+0,01237416
17	$(\mathrm{tg}\,u_m)_0$	+7,6841547	+1,44145128
18	$(\sin u_m)_0$	+0,99163813	+0,82163902
19	$(\mathrm{tg}\,u_m)_I$	+7,7126428	+1,44678615
20	$(\sin u_m)_I$	+0,99169901	+0,82262335
21	$(\mathrm{tg}\,u_m)_{II}$	+7,7126461	+1,44679833
22	$(\sin u_m)_{II}$	+0,99169902	+0,82262559
23	$(\mathrm{tg}\,u_m)_{III}$	+7,7126461	+1,44679837
24	$\cos u_m$	+0,12858090	+0,56858344
25	$\sin u_m$	+0,99169902	+0,82262560
26	x_m	+ 756 576,8	—1677 621,0
27	y_m	— 316 526,5	—3215 207,4
28	z_m	+6304 094,8	+5229 318,2
29	d^2	960436678	967436547
30	d	309909,1	311036,4
31	$(2l)^2$	154788860	154863313
32	$2l$	3934321,6	3935267,6
33	l	1967160,8	1967633,8
34	$\frac{d}{l}=\mathrm{tg}\,\frac{Q}{2}$	0,15754132	0,15807636
35	$\frac{Q}{2}$	8°57′10″,328	8°58′58″,006
36	Q°	17°54′20″,656	17°57′56″,012
37	Q	0,31251407	0,31353815
38	$\sin Q$	0,30745191	0,30844525
39	$\frac{Q}{\sin Q}$	1,01646488	1,01657637
40	S_I, S_{II}	3999099,7	4000500,1
41	$S_0 = S_I + S_{II}$	= 7999599,8	
	Exact val. S_0	= 7999605,5	
	ΔS_0	= — 5,7 м	

Calculation of azimuths A_{lm} and A_{ml}

For portion S_I / For portion S_{II}

		For portion S_I	For portion S_{II}
42	$x_2 - x_1$	−2407535,8	−2224333,2
43	$y_2 - y_1$	−3108896,0	−2406255,5
44	$z_2 - z_1$	+131996,4	−2179130,0

45	$\sin B_1$	$+0,93337178$		$+0,95401976$
46	$\cos B_1$	$+0,35891102$		$+0,29974297$
47	$\sin L_1$	$+0,54585826$		$-0,96764919$
48	$\cos L_1$	$+0,83787754$		$-0,25229951$
49	$(x_2-x_1)\cos L_1+(y_2-y_1)\sin L_1$	$-3714236,7$		$+2889609,4$
50	Numerator	$+3514138,7$		$-3409923,4$
51	Denominator	$-1290700,8$		$-1545277,1$
52	$\mathrm{ctg}\,\overline{A}_{1m}$	$-2,7226594$	$\mathrm{ctg}\,\overline{A}_{m2}$	$+2,2066744$
53	$\overline{A}_{1m}$	339°49′56″,368	$\overline{A}_{m2}$	204°22′43″,093
54	$\sin B_2$	$+0,95401976$		$+0,61271718$
55	$\cos B_2$	$+0,29974297$		$+0,79068957$
56	$\sin L_2$	$-0,96764919$		$-0,84401605$
57	$\cos L_2$	$-0,25229951$		$-0,53631791$
58	$(x_1-x_2)\cos L_2+(y_1-y_2)\sin L_2$	$+3615740,8$		$+3223868,0$
59	Numerator	$-3409923,2$		$-3696773,1$
60	Denominator	$+1545277,1$		$-586855,0$
61	$\mathrm{ctg}\,\overline{A}_{m1}$	$+2,2066743$	$\mathrm{ctg}\,\overline{A}_{2m}$	$+6,2992104$
62	$\overline{A}_{m1}$	24°22′43″,097	$\overline{A}_{2m}$	9°01′13″,564

Example 3.

u_1	+ 20°
L_1	0
u_2	− 20°
L_2	0

We find directly on the basis of geometric considerations:

$$\frac{d}{l}=\frac{a}{b}\cdot \mathrm{tg}\,\frac{u_1}{2}=\frac{1}{\sqrt{1-e^2}}\cdot \mathrm{tg}\,\frac{u_1}{2}=\mathrm{tg}\,Q=0,1769\ 2007$$

Q°	10°01′58″,633
Q	0,17510807
$2Q^\circ$	20°03′57″,266
$\sin 2Q^\circ$	0,34310084
$l=b\sin u$	2174175,2
S_0	4438527,4
Exact val. S_0	4438520,2 м
ΔS_0	+ 7,2 м

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