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OPTIMUM CONTOUR OF HEAT REJECTING TRIANGULAR FINS WITH MUTUAL IRRADIATION BETWEEN FINS AND COOLED BASE SURFACES. THE SYSTEM OF PARALLEL STAR - SHAPED RADIATORS. EFFECTS OF THE THERMAL RESISTANCE OF COATINGS

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In the present paper the results of the theoretical investigation of the star-shaped radiators are discussed.

The paper consists of the three parts. The first part deals with the one-dimensional problem of the optimum size and number of heat rejection triangular fins radially arranged at the apices of the polyhedral prism with mutual irradiation of the fins and the surface of the cooled prism.

The numerical solution of the problem is obtained with the assumption that the temperature gradient across the fin is negligible in comparison with that of along the fin and surroundings are assumed as a black body of zero temperature. The relationships of dimensionless parameters are given which determine the optimum (with respect to weight) number of fins and their geometry for any combination of the cooled prism temperature and size, amount of the rejected heat, the emissivity of radiating fins and heat conductivity of the fin material.

It is found that the optimum (in the sense of weight) number of radiating fins increases from 4 for black emitting surfaces to about 10-11 for the emissivity of these surfaces of the order of 0.5.

In the second part the thermal radiation characteristics of the infinite system of the four-fin-star-shaped coplanar radiators are calculated taking into account the mutual irradiation of all radiator elements. The problem is solved numerically under the same assumptions as in the first part. The relationships of the dimensionless parameters are given which allow to calculate all the optimal characteristics of the system for any distance between the adjacent radiators of the system. In the third part the problem is discussed taking into consideration the effect of the thermal resistance of the coating which may be used for increasing the emissivity of the radiating surfaces on the thermal radiation characteristics of the four-fin-star-shaped radiators. In addition to the above assumptions it was suggested that the heat rates along the fin in the coating are negligible in comparison with the heat rates along the fin. The given relationships of the dimensionless parameters describe the effectiveness of the coating for any heat conductivity of the latter and the radiator parameters.

§ I. Optimum Contour of Heat Rejecting Triangular Fins with Mutual Irradiation between Fins and Cooled Base Surfaces

Let us consider the one-dimensional problem of determination of the optimum size and number of heat rejecting triangular fins radially arranged at the apices of the polyhedral prism taking into account mutual irradiation of the fins and the base surface (Fig.1). In the case of negligible base-surface size such a problem was discussed for the fins of the optimum (Ref.1) and triangular contour (Ref.2).

We shall deal with the thin fins for which the law of thermal radiation and the equation of thermal conduction along a fin are true in the form of:

$$\frac{1}{2} Q(x) = -\lambda (L - x) \frac{\alpha}{2} \frac{dT}{dx} \qquad (1)$$

$$\frac{1}{2} dQ(x) = -q(x)\, dx \qquad (2)$$

where $Q(x)$ - the heat flow through the fin section with x-position
λ - the fin material thermal conductivity coefficient.
α - the angle between the side faces of a fin
$q(x)\,dx$ - the resultant radiation from the fin surface element.

Equations (1) and (2) lead to the following expression for the determination of temperature distribution along a fin:

$$(L - x) \frac{d^2T}{dx^2} - \frac{dT}{dx} - \frac{q(x)}{\lambda \frac{\alpha}{2}} = 0 \qquad (3)$$

Consider first the trapezoidal fins of length L_1 (Fig.1). In this case the boundary conditions for equation (3) may be written as

$$a)\ T = T_0 \text{ at } x = 0, \quad b)\ \frac{dT}{dx} = 0 \text{ at } x = L_1 \qquad (4)$$

When determining $q(x)$ we shall assume that the fin surface radiating element is at the central plane and the parameters of the surroundings $\mathcal{E}_{sur} = 1$ and $T_{sur} = 0$. Then the expression for $q(x)$ will have the form of (Ref.3)

$$q(x) = B(x) - H_{z \to x}(x) - H_{y \to x}(x) =$$

$$= B(x) - \frac{1}{2} \int_0^{L_1} B(z) \frac{(z_0 + z)(z_0 + x) \sin^2 \gamma \, dz}{[(z_0 + z)^2 + (z_0 + x)^2 - 2(z_0 + z)(z_0 + x)\cos\gamma]^{3/2}} \qquad (5)$$

$$- \frac{1}{2} \int_0^{y_{max}} B_1(y) \frac{x y \cos^2 \gamma/2 \, dy}{(x^2 + y^2 + 2xy \sin \gamma/2)^{3/2}},$$

where $\gamma = \frac{2\pi}{n}$ is the angle between the central planes of two adjacent fins.

In equation (5) $B(x)$, $B(z)$, $B_1(y)$ - the effective radiant flows per unit area and time leaving the fin surfaces "x", "z" and the prism surfaces "y" are determined by the following relations:

$$B(x) = \varepsilon\sigma T^4(x) + (1-\varepsilon)[H_{z\to x}(x) + H_{y\to x}(x)] \qquad (6)$$

$$B(z) = \varepsilon\sigma T^4(z) + (1-\varepsilon)[H_{x\to z}(z) + H_{y\to z}(z)] \qquad (7)$$

$$B_1(y) = \varepsilon\sigma T_0^4 + (1-\varepsilon)[H_{x\to y}(y) + H_{z\to y}(y)] \qquad (8)$$

where $H_{z\to x}(x)$, $H_{y\to x}(x)$; $H_{x\to z}(z)$, $H_{y\to z}(z)$ - are the heat flows transfered from the adjacent fin and the prism surfaces to the fin which is of interest;

$H_{x\to y}(y)$, $H_{z\to y}(y)$ - are the heat flows from the fins to the prism surface.

The relations determining $H_{z\to x}(x)$, $H_{y\to x}(x)$ are evident from equation (5). The heat flow $H_{y\to z}(z)$ is determined by the following expression:

$$H_{y\to z}(z) = \frac{1}{2}\int_{\Delta}^{y_1-\Delta} B_1(y)\,\frac{z\,(y_1-y)\cos^2\frac{\gamma}{2}\,dy}{[(y_1-y)^2+z^2+2z(y_1-y)\sin\gamma/2]^{3/2}} \qquad (9)$$

The heat flow relations $H_{x\to z}(z)$; $H_{x\to y}(y)$; $H_{z\to y}(y)$ are obtaind from the expressions for $H_{z\to x}(x)$, $H_{y\to x}(x)$, $H_{y\to z}(z)$, provided that their variables are interchanged.

Introduce the following dimensionless variables:

$$\bar{T} = \frac{T}{T_0} ; \bar{x} = \frac{x}{L_1} ; \bar{z} = \frac{z}{L_1} ; \bar{y} = \frac{y}{L_1} ;$$
$$\bar{z}_0 = \frac{z_0}{L_1} ; \bar{L} = \frac{L}{L_1} ; \bar{H} = \frac{H}{\sigma T_0^4} ; \qquad (10)$$

After integration with the use of the second boundary condition, the equation (5) in these variables will be written as

$$(\bar{L}-\bar{x})\frac{d\bar{T}}{d\bar{x}} + N\int_{\bar{x}}^{1}[\bar{B}(\bar{x}) - \bar{H}_{z\to x}(\bar{x}) - \bar{H}_{y\to x}(\bar{x})]\,d\bar{x} = 0 \qquad (11)$$

$$N = \frac{L_1\sigma T_0^3}{\lambda\frac{\alpha}{2}} \qquad (12)$$

$$\bar{B}(\bar{x}) = \varepsilon\bar{T}^4(\bar{x}) + (1-\varepsilon)[\bar{H}_{z\to x}(\bar{x}) + \bar{H}_{y\to x}(\bar{x})] \qquad (13)$$

$$\bar{B}_1(\bar{y}) = \varepsilon + (1-\varepsilon)[\bar{H}_{z\to y}(\bar{y}) + \bar{H}_{x\to y}(\bar{y})] \qquad (14)$$

$$\bar{H}_{z\to x}(\bar{x}) = \frac{1}{2}\int_{0}^{1}\bar{B}(\bar{z})\,\frac{(\bar{z}_0+\bar{z})(\bar{z}_0+\bar{x})\sin^2\gamma\,d\bar{z}}{[(\bar{z}_0+\bar{z})^2+(\bar{z}_0+\bar{x})^2-2(\bar{z}_0+\bar{z})(\bar{z}_0+\bar{x})\cos\gamma]^{3/2}} \qquad (15)$$

$$\bar{H}_{y\to x}(\bar{x}) = \frac{1}{2}\int_{\Delta}^{\bar{y}_1-\Delta}\bar{B}_1(\bar{y})\,\frac{\bar{x}\,\bar{y}\cos^2\gamma/2\,d\bar{y}}{(\bar{x}^2+\bar{y}^2+2\bar{x}\bar{y}\sin\frac{\gamma}{2})^{3/2}} \qquad (16)$$

$$\bar{H}_{x\to y}(\bar{y}) = \frac{1}{2}\int_{0}^{1}\bar{B}(\bar{x})\,\frac{\bar{x}\,\bar{y}\cos^2\gamma/2\,d\bar{x}}{(\bar{x}^2+\bar{y}^2+2\bar{x}\bar{y}\sin\gamma/2)^{3/2}} \qquad (17)$$

$$\bar{H}_{z\to y}(\bar{y}) = \frac{1}{2}\int_{0}^{1}\bar{B}(\bar{z})\,\frac{\bar{z}\,(\bar{y}_1-\bar{y})\cos^2\gamma/2\,d\bar{z}}{[\bar{z}^2+(\bar{y}_1-\bar{y})^2+2\bar{z}(\bar{y}_1-\bar{y})\sin\gamma/2]^{3/2}} \qquad (18)$$

Due to symmetry of the problem, $\bar{B}(\bar{x}) \equiv \bar{B}(\bar{z})$ at $\bar{x} = \bar{z}$ The boundary condition for equation (11) will be $\bar{T} = 1{,}0$ at $\bar{x} = 0$

According to equation (5) the heat loss from the surfaces of one fin will be written in the dimensionless variables as

$$q_{fin} = 2L_1\sigma T_0^4 \int_0^1 [\bar{B}(\bar{x}) - \bar{H}_{2\to x}(\bar{x}) - \bar{H}_{y\to x}(\bar{x})]\, d\bar{x} \qquad (19)$$

Since resultant heat flow radiated from the surface of the cooled prism equals to

$$q_1(y) = B_1(y) - H_{x\to y}(y) - H_{2\to y}(y) \qquad (20)$$

the amount of the heat in the dimensionless variables radiated from prism surface will be determined by the following expression:

$$q_{pr} = L_1\sigma T_0^4 \int_{\bar{\delta}}^{\bar{y}_1-\bar{\delta}} [\bar{B}_1(\bar{y}) - \bar{H}_{x\to y}(\bar{y}) - \bar{H}_{2\to y}(\bar{y})]\, d\bar{y} \qquad (21)$$

Let us find the emission coefficient ζ of the system which is the ratio of the heat $(q_{fin} + q_{pr})n$ radiated by the system to the heat q_{ideal} which would be radiated by this system if the fin material conductivity were infinitely high and ε were one (R=1.5).

The expression for q_{ideal} has the form

$$q_{ideal} = 2n(L_1 + z_0)\sigma T_0^4 \sin\gamma/2 \qquad (22)$$

and the emission coefficient is determined by

$$\zeta = \frac{1}{2(1+\bar{z}_0)\sin\gamma/2}\left\{2\int_0^1 \bar{q}_{res}(\bar{x})\,d\bar{x} + \int_{\bar{\delta}}^{\bar{y}_1-\bar{\delta}} \bar{q}_{1res}(\bar{y})\,d\bar{y}\right\} \qquad (23)$$

from which it is seen that this coefficient is a function of the parameters $N, \gamma, \bar{z}_0$ and ε. Thus, the heat radiated by the system equals to

$$q_\Sigma = \zeta q_{ideal} = 2n\zeta L_1(1+\bar{z}_0)\sigma T_0^4 \sin\gamma/2. \qquad (24)$$

The solution of the set of equations (11) through (18) and (23) for the triangular fins was derived numerically as an asymptotical one at $\bar{L} \to 1,0$ and $\bar{\Delta} \to 0$

The relationships between the emission coefficient ζ and the dimensionless thermal conduction parameter N are given in figures 2,3,4 and 5 for various values of the parameters γ, $\bar{z}_0$ and ε.

By using the relationships of figures 2 through 5 and having a number of fins and ε selected, the optimum geometry of triangular fins can be easy determined provided that the temperature T_0, the prism size $\bar{z}_0$ and the amount of the radiated heat are known. Really, using the relation-ships $\zeta = \zeta(N)$ for selected n, ε and $\bar{z}_0$ values, one can determine α from expressions (12) and (24) for a number of L_1 values and, thus, evaluate the minimum area of a fin. This calculation results are represented in figures 6,7 and 8 where $\bar{F}$ and $\bar{z}_0^*$ are the dimensionless variables, related with

physical magnetudes by the following expressions:

$$\bar{F} = \frac{F}{F_{opt}} \; ; \qquad \bar{\varepsilon}_o^* = \frac{\varepsilon_o}{\frac{q_\Sigma}{\varepsilon \sigma T_o^4}} \; , \qquad (25)$$

where F is the total cross-section area of the system of the optimum triangular fins, $F_{opt} = \frac{q_\Sigma^3}{\lambda \sigma^2 \varepsilon^2 T_o^3}$ is the cross-section area of the optimum fin of minimum weight (Ref.4).

In figure 9 the relationships between N_{opt} and $\bar{\varepsilon}_o^*$ are given. Using them together with the figures 6, 7, 8 the optimum fin geometry can be determined immediately.

The relations $\bar{F} = \bar{F}(n)$ at $\bar{\varepsilon}_o^* = 0$, represented in figures 6, 7 and 8, correspond to the case of the negligible prism size.

In accordance with equation (24), at $\bar{\varepsilon}_o^* = \frac{1}{2n \sin \delta/2}$ a complete degeneration of fins occurs (in this case $\bar{F} = 0$ and $\bar{\varepsilon}_o = \infty$).

It is evident that equally with increasing the optimum fin number when the emissivity diminishes the optimum fin number increases with the increase of the parameter $\bar{\varepsilon}_o^*$

The dotted curve for the system of the optimum fins equally spaced around a base prism is given in figure 6 (Ref.1). It is seen that at $\bar{\varepsilon}_o^* = 0$ and $\varepsilon = 1,0$ the change of the optimum fins by the triangular fins results in the increase of the fin cross-section by about 6 per cent.

§ 2. Theoretical Investigation of Thermal Radiation Characteristics of Infinite Parallel Star-Shaped Radiator System

Consider the problem of calculation of the thermal radiation characteristics of the infinite radial coplanar radiator system having the heat rejecting triangular fins (Fig.10).

The mathematical model of the radiator system section is shown in figure 10a with the main geometrical parameters.

The governing equation for the problem solution are the energy conservation equations for the arbitrary elements dV_1, dV_2 of the transversal and longitudinal radiator fins (fig.10b)

$$(l - x)\frac{d^2 T_1}{dx^2} - \frac{dT_1}{dx} - \frac{q_{1res}(x)}{\lambda \cdot \delta/2} = 0 \qquad (26)$$

$$(l - z)\frac{d^2 T_2}{dz^2} - \frac{dT_2}{dz} - \frac{q_{2res}(z)}{\lambda \cdot \delta/2} = 0 \; , \qquad (27)$$

where $q_{1res}(x)$, $q_{2res}(z)$ are the resulting radiation flows from the side faces dS_1, dS_2 of the elements dV_1, dV_2

First, consider the trapezoidal fins (fig.10b). In this case one can write the following boundary conditions for equations (26), (27):

$$T_1 = T_0 \quad at \quad x = 0 \tag{28}$$

$$\frac{dT_1}{dx} = 0 \quad at \quad x = L_1 \tag{29}$$

$$T_2 = T_0 \quad at \quad z = 0 \tag{30}$$

$$\frac{dT_2}{dz} = 0 \quad at \quad z = L_1 \tag{31}$$

According to the method of the resulting radiation, expressions describing the flows of the resulting radiation $q_{1res}(x)$, $q_{2res}(z)$ have the following form:

$$q_{1res}(x) = B_1(x) - H_{z\to x}(x) - H_{y\to x}(x) - H_{i\to x}(x) - H_{j\to x}(x) - H_{\kappa\to x}(x) \tag{32}$$

$$q_{2res}(z) = B_2(z) - H_{x\to z}(z) - H_{y\to z}(z) - H_{j\to z}(z) - H_{\kappa\to z}(z) \tag{33}$$

where

$$B_1(x) = \varepsilon\sigma T_1^4(x) + (1-\varepsilon)[H_{y\to x}(x) + H_{z\to x}(x) + H_{\kappa\to x}(x) + H_{j\to x}(x) + H_{i\to x}(x)] \tag{34}$$

is the flow of the effective radiation from the side face dS_1 of the element dV_1,

$$B_2(z) = \varepsilon\sigma T_2^4(z) + (1-\varepsilon)[H_{x\to z}(z) + H_{y\to z}(z) + H_{\kappa\to z}(z) + H_{j\to z}(z)] \tag{35}$$

is the flow of the effective radiation from the side face dS_2 of the element dV_2 (Fig. 10b),

$H_{z\to x}(x)$; $H_{y\to x}(x)$; $H_{i\to x}(x)$; $H_{j\to x}(x)$; $H_{\kappa\to x}(x)$

are the total heat flows radiated from the surfaces denoted by the first letter of the index to the surface dS_1, $H_{x\to z}(z)$; $H_{y\to z}(z)$; $H_{j\to z}(z)$; $H_{\kappa\to z}(z)$ are the total heat flows radiated from the surfaces "x"; „y"; „j"; „κ" to the elemental surface dS_2

Similarly, the flow of resulting radiation leaving the base surface equals to

$$q_{3res}(y) = B_3(y) - H_{x\to y}(y) - H_{z\to y}(y) - H_{\kappa\to y}(y) - H_{j\to y}(y) - H_{i\to y}(y) \cdots , \tag{36}$$

where

$$B_3(y) = \varepsilon\sigma T_0^4 + (1-\varepsilon)[H_{x\to y}(y) + H_{z\to y}(y) + H_{\kappa\to y}(y) + H_{j\to y}(y) + H_{i\to y}(y)] \tag{37}$$

is the flow of the effective radiation from the base surface and

$H_{x\to y}(y)$; $H_{z\to y}(y)$; $H_{\kappa\to y}(y)$; $H_{j\to y}(y)$; $H_{i\to y}(y)$

are the total heat flows radiated from the surfaces, denoted by the first letter of the index, to the base surface.

Integration of equations (26) and (27) using the boundary conditions (29), (31) and the dimensionless variables (10) yields the following set of the equations for the radiator:

$$(\bar{L}-\bar{x})\frac{d\bar{T}_1}{d\bar{x}} + N\int_{\bar{x}}^{1}\bar{q}_{1res}(\bar{x})\,d\bar{x} = 0 \qquad (20a)$$

$$(\bar{L}-\bar{z})\frac{d\bar{T}_2}{d\bar{z}} + N\int_{\bar{z}}^{1}\bar{q}_{2res}(\bar{z})\,d\bar{z} = 0 \,, \qquad (21a)$$

where

$$N = \frac{\lambda_1 \sigma T_0^3}{\lambda \cdot \frac{d}{2}} \qquad (12a)$$

$$q_{1res}(\bar{x}) = \bar{B}_1(\bar{x}) - H_{2\to x}(\bar{x}) - \bar{H}_{y\to x}(\bar{x}) - H_{i\to x}(\bar{x}) - \bar{H}_{j\to x}(\bar{x}) - \bar{H}_{k\to x}(\bar{x}) \qquad (22a)$$

$$\bar{q}_{2res}(\bar{z}) = \bar{B}_2(\bar{z}) - \bar{H}_{x\to 2}(\bar{z}) - \bar{H}_{y\to 2}(\bar{z}) - \bar{H}_{j\to 2}(\bar{z}) - \bar{H}_{k\to 2}(\bar{z}) \qquad (33a)$$

$$\bar{B}_1(\bar{x}) = \varepsilon \bar{T}_1^4(\bar{x}) + (1-\varepsilon)[\bar{H}_{y\to x}(\bar{x}) + \bar{H}_{2\to x}(\bar{x}) + \bar{H}_{k\to x}(\bar{x}) + \bar{H}_{j\to x}(\bar{x}) + \bar{H}_{i\to x}(\bar{x})] \qquad (34a)$$

$$\bar{B}_2(\bar{z}) = \varepsilon_1 \bar{T}_2^4(\bar{z}) + (1-\varepsilon)[\bar{H}_{x\to 2}(\bar{z}) + \bar{H}_{y\to 2}(\bar{z}) + \bar{H}_{k\to 2}(\bar{z}) + \bar{H}_{j\to 2}(\bar{z})] \qquad (35a)$$

$$\bar{H}_{2\to x}(\bar{x}) = \frac{1}{2}\int_0^1 \bar{B}_2(\bar{z})\frac{(\bar{z}_0+\bar{z})(\bar{z}_0+\bar{x})\,d\bar{z}}{[(\bar{z}_0+\bar{z})^2+(\bar{z}_0+\bar{x})^2]^{3/2}} \qquad (36)$$

$$\bar{H}_{y\to x}(\bar{x}) = \frac{1}{4}\int_{\bar{\Delta}}^{\bar{y}_1-\bar{\Delta}} \bar{B}_3(\bar{y})\frac{\bar{x}\,\bar{y}\,d\bar{y}}{(\bar{x}^2+\bar{y}^2+\bar{x}\bar{y}\sqrt{2})^{3/2}} \qquad (39)$$

$$\bar{y}_1 = \bar{j}_1 = \bar{z}_0\sqrt{2}\,; \quad \bar{z}_0 = \frac{z_0}{\lambda_1}\,; \quad \bar{\Delta} = \frac{\Delta}{\lambda_1} \qquad (40)$$

$$\bar{H}_{k\to x}(\bar{x}) = \frac{1}{2}\int_0^1 \bar{B}_1(\bar{k})\frac{(2\bar{z}_0+\bar{e}+2)^2\,d\bar{k}}{[(\bar{k}-\bar{x})+(2\bar{z}_0+\bar{e}+2)^2]^{3/2}} \qquad (41)$$

$$\bar{H}_{i\to x}(\bar{x}) = \frac{1}{2}\int_0^1 \bar{B}_2(\bar{i})\frac{(\bar{z}_0+\bar{x})(\bar{z}_0+\bar{e}+2-\bar{i})\,d\bar{i}}{[(\bar{z}_0+\bar{x})^2+(\bar{z}_0+\bar{e}+2-\bar{i})^2]^{3/2}} \qquad (42)$$

$$\bar{H}_{j\to x}(\bar{x}) = \frac{1}{\sqrt{2}}\int_{\bar{\Delta}}^{\bar{j}_1-\bar{\Delta}} \bar{B}_3(\bar{j})\frac{[(2\bar{z}_0+\bar{e}+2)\sqrt{2}-\bar{j}\,](2\bar{z}_0+\bar{e}+2+\bar{x})\,d\bar{j}}{\{(\bar{x}\sqrt{2}+\bar{j})^2+[(2\bar{z}_0+\bar{e}+2)\sqrt{2}-\bar{j}]^2\}^{3/2}} \qquad (43)$$

$$\bar{H}_{x\to 2}(\bar{z}) = \frac{1}{2}\int_0^1 \bar{B}_1(\bar{x})\frac{(\bar{z}_0+\bar{x})(\bar{z}_0+\bar{z})\,d\bar{x}}{[(\bar{z}_0+\bar{x})^2+(\bar{z}_0+\bar{z})^2]^{3/2}} \qquad (44)$$

$$\bar{H}_{y\to 2}(\bar{z}) = \frac{1}{4}\int_{\bar{\Delta}}^{\bar{y}_1-\bar{\Delta}} \bar{B}_3(\bar{y})\frac{\bar{z}(\bar{y}_1-\bar{y})\,d\bar{y}}{[\bar{z}^2+(\bar{y}_1-\bar{y})^2+\bar{z}(\bar{y}_1-\bar{y})\sqrt{2}]^{3/2}} \qquad (45)$$

$$\bar{H}_{k\to 2}(\bar{z}) = \frac{1}{2}\int_0^1 \bar{B}_1(\bar{k})\frac{(\bar{z}_0+\bar{k})(\bar{z}_0+\bar{e}+2-\bar{z})\,d\bar{k}}{[(\bar{z}_0+\bar{k})^2+(\bar{z}_0+\bar{e}+2-\bar{z})^2]^{3/2}} \qquad (46)$$

$$\bar{H}_{j\to 2}(\bar{z}) = \frac{1}{2\sqrt{2}}\int_{\bar{\Delta}}^{\bar{j}_1-\bar{\Delta}} \bar{B}_3(\bar{j})\frac{(\bar{z}_0-\frac{\bar{j}}{\sqrt{2}})(2+\bar{e}-\bar{z})\,d\bar{j}}{[(\bar{z}_0-\frac{\bar{j}}{\sqrt{2}})^2+(\bar{z}_0+\bar{e}+2-\frac{\bar{j}}{\sqrt{2}}-\bar{z})^2]^{3/2}} \qquad (47)$$

$$\bar{B}_3(\bar{y}) = \varepsilon + (1-\varepsilon)[\bar{H}_{x\to y}(\bar{y}) + \bar{H}_{2\to y}(\bar{y}) + \bar{H}_{x\to y}(\bar{y}) + \bar{H}_{j\to y}(\bar{y}) + \bar{H}_{i\to y}(\bar{y})] \quad (37a)$$

$$\bar{H}_{x\to y}(\bar{y}) = \frac{1}{4}\int_0^1 \bar{B}_1(\bar{x}) \frac{\bar{x}\,\bar{y}\,d\bar{x}}{(\bar{x}^2+\bar{y}^2+\bar{x}\,\bar{y}\sqrt{2})^{3/2}} \quad (48)$$

$$\bar{H}_{2\to y}(\bar{y}) = \frac{1}{4}\int_0^1 \bar{B}_2(\bar{z}) \frac{\bar{z}(\bar{y}_1-\bar{y})d\bar{z}}{[\bar{z}^2+(\bar{y}_1-\bar{y})^2+\bar{z}(\bar{y}_1-\bar{y})\sqrt{2}\,]^{3/2}} \quad (49)$$

$$\bar{H}_{x\to y}(\bar{y}) = \frac{1}{2}\int_0^1 \bar{B}_1(\bar{x}) \frac{(2+2\bar{z}_0+\bar{e}+\bar{x})[4+2\bar{z}_0+2\bar{e}+(\bar{y}_1-\bar{y})\sqrt{2}]d\bar{x}}{\{(2+2\bar{z}_0+\bar{e}+\bar{x})^2+[(2+\bar{e}-\bar{x})+(\bar{y}_1-\bar{y})\sqrt{2}]^2\}^{3/2}} \quad (50)$$

$$\bar{H}_{j\to y}(\bar{y}) = \frac{1}{\sqrt{2}}\int_{\bar{\delta}}^{1-\bar{\delta}} \bar{B}_3(\bar{j}) \frac{[(2+\bar{e})+(\bar{y}_1-\bar{y})\sqrt{2}]\cdot[(2+\bar{e})+(\bar{y}_1-\bar{j})\sqrt{2}]d\bar{j}}{\{[(2+\bar{e})+(\bar{y}_1-\bar{y})\sqrt{2}]^2+[(2+\bar{e})+(\bar{y}_1-\bar{j})\sqrt{2}]^2\}^{3/2}} \quad (51)$$

$$\bar{H}_{i\to y}(\bar{y}) = \frac{1}{\sqrt{2}}\int_0^1 \bar{B}_2(\bar{i}) \frac{(\bar{y}_1-\bar{y})(2+\bar{e}-\bar{i})\,d\bar{i}}{\{[(2+\bar{e}-\bar{i})+(\bar{y}_1-\bar{y})\sqrt{2}]^2+(2+\bar{e}-\bar{i})^2\}^{3/2}} \quad (52)$$

The boundary conditions for equations (26a) and (27a) are

$$\bar{T}_1 = 1.0 \qquad at \quad \bar{x} = 0 \quad (26a)$$

$$\bar{T}_2 = 1.0 \qquad at \quad \bar{z} = 0 \quad (30a)$$

The expressions for the total heat flows which are concluded into the above set of the equations have been obtained on the basis of the formula for the elemental heat flow radiated from the infinitely thin and long strip $d\bar{z}$ to elemental area dS (Fig.10a), (Ref.3). $dH = B\frac{d\bar{s}\sin\varphi}{2}$ assuming that radiating elements of the fin surfaces are their central planes.

The set of equations (26a), (27a), (34a), (35a), (37a) together with the boundary conditions (28a), (30) and formulas (32a), (33a), (36) through (52) determines the problem solution completely.

From the results of the problem solution the total radiator-section heat loss data are of the most practical interest. The amount of the heat radiated from one face of a transversal fin (of unit width) is determined by the following expression:

$$Q_{1fin} = L_1\sigma T_0^4 \int_0^1 \bar{q}_{1res}(\bar{x})\,d\bar{x} \quad (53)$$

Similarly, the heat loss from one face of a longitudinal fin equals to

$$Q_{2fin} = L_1\sigma T_0^4 \int_0^1 \bar{q}_{2res}(\bar{z})\,d\bar{z} \quad (54)$$

According to equation (36) the amount of the heat, radiated my the prism surface (of unit width) will be written as

$$Q_{3pr} = L_1\sigma T_0^4 \int_{\bar{\delta}}^{1-\bar{\delta}} \bar{q}_{3res}(\bar{y})\,d\bar{y} \quad (55)$$

The overall heat loss from a section of the radiator of the unit width equals to

$$Q_\Sigma = L_1\sigma T_0^4 \left[\int_0^1 \bar{q}_{1res}(\bar{x})\,d\bar{x} + \int_0^1 \bar{q}_{2res}(\bar{z})\,d\bar{z} + \int_{\bar{\delta}}^{1-\bar{\delta}} \bar{q}_{3res}(\bar{y})\,d\bar{y}\right] \quad (56)$$

To compare the individual radiator section effectiveness with the effectiveness of a similar section of the radiator system, let us express an emission coefficient by the relation

$$\zeta = \frac{Q_\Sigma}{Q_{ideal}}, \quad (57)$$

where

$$Q_{ideal} = \sqrt{2}\,\sigma T_0^4 L_1 (1+\bar{z}_0) \qquad (58)$$

According to formula (56) one obtains

$$\zeta = \frac{\int_0^1 \bar{q}_{1res}(\bar{x})\,d\bar{x} + \int_0^1 \bar{q}_{2res}(\bar{z})\,d\bar{z} + \int_{\bar{\delta}}^{\bar{y}_1-\bar{\delta}} \bar{q}_{3res}(\bar{y})\,d\bar{y}}{\sqrt{2}(1+\bar{z}_0)} \qquad (57a)$$

It follows from formula (57a) and the set of equations (20a), (27a), (32a) through (52) that the emission coefficient is a function of the parameters $N, \bar{z}_0, \varepsilon, \bar{\ell}, \bar{\Delta}, \bar{L}$ At $\bar{\ell} \to \infty$ the emission coefficient ζ determines the efficiency of the single radiator section considered in the first section of this paper (§ 1).

If the relations $\zeta(N, \bar{z}_0, \varepsilon, \bar{\ell}, \bar{\Delta}, \bar{L})$ are known the overall heat loss from the radiator section is determined by

$$Q_\Sigma = \sqrt{2}(1+\bar{z}_0)\,\zeta\, L_1 \sigma \bar{T}_0^4 \qquad (56a)$$

The solution for the case of the triangular heat rejecting fins has been derived as an asymptotical one with $\bar{L} = \frac{L}{L_1} \to 1{,}0$ and $\bar{\Delta} \to 0$ from the set of equations (20a), (27a), (32a) through (52) together with the boundary conditions (28a), (30a). The solution has been carried out numerically for the general case of the arbitrary distance between the adjacent radiators of the system ($\bar{\ell} = 0 \div \infty$) in the range of emissivity values $\varepsilon = 1.0 \div 0.75$.

The emission coefficients are plotted in fig.11 through 15 as a function of the dimensionless heat conductivity parameter for the various values of the parameters $\bar{z}_0, \varepsilon, \bar{\ell}$

The relationships $\zeta(N)$ at $\bar{\ell} > 10$ practically agree with the similar relations for the individual radiator, considered in the first section.

Using the relationships given in the figures 11 through 15 and the specified values of ε and $\bar{\ell}$, one can find the optimum triangular fins geometry, provided that the fin base temperature T_0, the overall heat loss from one section Q_Σ and the prism "radius" z_0 are known.

When determining the optimum geometry in the case of the following calculation, procedure may be ordered. Setting a number of the values of $\bar{z}_0 = \frac{z_0}{L_1}$ one finds corresponding L_1 values and ζ values, determined by formula (56a)

Furthermore, using the relationships of figures 11 through 15[x] we determine the values of the dimensionless heat conductivity parameter N. From formula 12a determine the values of the angle α.

---

x) The relationships $\zeta(N)$ for the intermediate $\bar{z}_0$ values can be obtained by the graphical interpolation of the curves, presented in figures 11 through 15.

Finally, the area F, the length L_1 and the angle are plotted against $\bar{z}_0$, therefrom the optimum fin geometry is determined. In the case of $\bar{z}_0 = 0$ the optimum fin geometry is calculated immediately for a number of N values with $\eta(N)$ being used (Figs.11 through 15). The results of such calculations of the total optimum triangular fin area are presented in figure 16 where $\bar{F}$ and $\bar{z}_0^*$ are the dimensionless parameters related with the physical magnitudes by expressions (25). For comparison the curves of the nondimensional minimum area of the triangular fins for the radiator system without the transversal heat rejecting fins are also shown in this figure in dotted lines for the values of the parameter $\bar{\ell} = 0;\ 2.0$ and $\varepsilon = 1.0;\ 0.875;\ 0.75$.

It is evident from the comparison of $\bar{F}$ ($\bar{z}_0^*$) curves that at $\bar{\ell} = 0$ the system of the radiators with the transversal heat rejecting fins is more advantageous than the system without such fins only when the emissivity $\varepsilon \leq 0.85 \div 0.9$. The result obtained in this particular case of the problem confirms G.Z.Grodzovsky's idea: that is, the weight characteristics of a usual panel radiator may be improved at the expense of additional transversal fins installed on the tube. When the dimensionless distance $\bar{\ell}$ is increased an advantage of the system with the transversal heat rejecting fins over the one without such fins becomes evident. In addition, the charts of figure 16 illustrate clearly the difference between the radiator system and the single radiator weight characteristics.

Relations between the emission coefficient η_1 and the dimensionless heat conductivity parameter N are shown in figure 17 for various values of the parameters $\bar{z}_0, \varepsilon$ and $\bar{\ell} = 0$ for the radiator system without the transversal heat rejecting fins with the longitudinal triangular fins. The emissivity coefficient η_1 is determined by the formula

$$\eta_1 = \int_0^1 \bar{q}_{eses}(\bar{z})\, d\bar{z} + \int_{\bar{\delta}}^{\bar{y}_1 - \bar{\delta}} \bar{q}_{sses}(\bar{y})\, d\bar{y} \qquad (59)$$

which expresses the ratio of the total amount of the heat, radiated by the radiator section of unit width (one side of the fin plus the cooled prism surface) to the ideal amount of the heat radiated by the black isothermal fin surface when the incident radiation is absent ($Q_{ideal} = 4\sigma T_0^4$).

§ 3. Influence of Coating Thermal Resistance on Heat Radiation Characteristics of Star-Shaped Radiators

Consideration of the coating thermal resistance effect on the radiation efficiency is of great practical interest for both evaluation of the coatings used and selection of the correction factors which must be introduced when theoretical investigation results are used without taking into consideration the thermal resistance of coatings.

The problem of taking into account the coating thermal resistance in the simplest case of the panel radiator with rectangular fins (Fig.18a) was discussed in the reference 6. In the present paper the problem of evaluation of the coating thermal resistance influence is discussed in the general case.

of radiation with mutual irradiation of all the construction elements (the heat rejecting triangular fins, the base surfaces) being taken into account (Fig.1bb); this enables more exact selection of the correct factor values.

The radiator scheme with the main geometrical parameters is shown in the figure 1c.

The problem solution is being developed in the one - dimensional statement (thin fins) under the following assumptions:

a) the heat rejecting fins of the radiator have an identical geometry

b) the fin material and coating thermophysical properties are independent of a temperature

λ = constant, λ_1 = constant, ε_1 = constant

c) the temperature of the cooled prism is uniform

T_o = constant, t_o = constant

d) the parameters of the surrounding $\varepsilon_{sur} = 1,0$; $T_{sur} = 0°K$

e) the radiating surfaces are grey diffusely reflecting surfaces

f) the coating temperature gradient in the direction of the fin length as compared to that in the direction of normal to the fin surface is negligible small.

The governing equations used in the solution of the problem are the ones describing law of energy conservation

$$\frac{1}{2} dQ_{accum} = dQ_{cond} \qquad (60)$$

$$dQ_{cond} = q_{res}(x)\, dS, \qquad (61)$$

where

dQ_{accum} is the amount of the heat accumulated in the element dV (Fig.1bb) due to net conduction

dQ_{cond} is the amount of the heat flowing through the coating area dS

dS is the side face of the element dV

$q_{res}(x)\, dS$ is the resultant radiation from the side face

By applying Fourier's law, we can easely transform equations (60), (61) to the form of the equations which describe the temperature profile along the radiator fin

$$\frac{\lambda_1}{\delta}\left[T(x) - t(x)\right] = q_{res}\left[t(x)\right] \qquad (61a)$$

$$(L - x)\frac{d^2T}{dx^2} - \frac{dT}{dx} - \frac{q_{res}\left[t(x)\right]}{\lambda \frac{\delta}{2}} = 0 \qquad (60a)$$

where

λ ; λ_1 are the thermal conductivity coefficient of the fin material and the coating, respectively.

δ is the coating thickness

$t(x)$ is the temperature of the coating radiating surface

$T(x)$ is the fin temperature.

As above, consider first the trapezoidal rins of L_1 length (Fig.1ca). In this case the following bonuary conditions ror equation (c0a) are taking:

$$T = T_0 \quad at \quad x = 0 \tag{02}$$

$$\frac{dT}{dx} = 0 \quad at \quad x = L_1 \tag{03}$$

Accoording to the metnod of the resultant radiation expression ior the resultant radiation rlow $q_{res}[t(x)]$ rrom the side race dS or the element dV is written as follows:

$$q_{res}(x) = B(x) - H_{z\to x}(x) - H_{y\to x}(x) \tag{04}$$

where

$$B(x) = \varepsilon_1 \sigma t^4(x) + (1-\varepsilon_1)\left[H_{z\to x}(x) + H_{y\to x}(x)\right] \tag{05}$$

is tne effective radiation rlow leaving tne side race dS or tne element dV.

$H_{z\to x}(x)$; $H_{y\to x}(x)$ are tne total neat rlows radiated rrom tne surraces, denoted by tne rirst letter or the index, to tne surrace dS.

When deriving tne rormulas for $H_{z\to x}(x)$; $H_{y\to x}(x)$ tne assumption is made that tne radiating element or tne rin surrace is in its central plane. Then (Ref.3).

$$H_{z\to x}(x) = \frac{1}{2}\int_0^{L_1} B(z)\frac{(z_0+z)(z_0+x)\sin^2\gamma\, dz}{\left[(z_0+z)^2+(z_0+x)^2-2(z_0+z)(z_0+x)\cos\gamma\right]^{3/2}} \tag{06}$$

$$H_{y\to x}(x) = \frac{1}{2}\int_0^{y_1} B_1(y)\frac{xy\cos^2\frac{\gamma}{2}\, dy}{(x^2+y^2+2xy\sin\frac{\gamma}{2})^{3/2}} \tag{07}$$

In equation (07)

$$B_1(y) = \varepsilon_1 \sigma t_0^4 + (1-\varepsilon_1)\left[H_{x\to y}(y) + H_{z\to y}(y)\right] \tag{08}$$

is tne rlow of tne effective radiation rrom tne base surrace. $H_{x\to y}(y)$; $H_{z\to y}(y)$ are tne total neat rlows radiated rrom tne rin surraces to tne base surraces or tne prism;

$$H_{x\to y}(y) = \frac{1}{2}\int_0^{L_1} B(x)\frac{xy\cos^2\frac{\gamma}{2}\, dx}{(x^2+y^2+2xy\sin\frac{\gamma}{2})^{3/2}} \tag{09}$$

$$H_{z\to y}(y) = \frac{1}{2}\int_0^{L} B(z)\frac{z(y_1-y)\cos^2\frac{\gamma}{2}\, dz}{\left[z^2+(y_1-y)^2+2z(y_1-y)\sin\frac{\gamma}{2}\right]^{3/2}} \tag{10}$$

It is convenient to transrorm tne set or equations (c0a), (c1a), (04) tnrough (10) together witn tne boundary conditions (02), (03) into tne dimensionless rorm by introducing tne nondimensional variables

$$\bar{x} = \frac{x}{L_1};\quad \bar{z} = \frac{z}{L_1};\quad \bar{y} = \frac{y}{L_1};\quad \bar{t} = \frac{t}{T_0}$$
$$\bar{T} = \frac{T}{T_0};\quad \bar{H} = \frac{H}{\sigma T_0^4};\quad \bar{B} = \frac{B}{\sigma T_0^4};\quad \bar{L} = \frac{L}{L_1} \tag{11}$$

As a result or tne transrormation and tne integration or equation (c0a) togetner witn tne boundary condition (03) tne set of tne equations will be

written in the following form:

$$(\bar{L}-\bar{x})\frac{d\bar{T}}{d\bar{x}} + N\int_{\bar{x}}^{1} \bar{q}_{res}\left[\bar{t}(\bar{x})\right] d\bar{x} = 0 \qquad (60b)$$

$$\bar{T} = \bar{t} + N_1\, \bar{q}_{res}\left[\bar{t}(\bar{x})\right] \qquad (61b)$$

where

$$N = \frac{\lambda_1 \sigma T_0^3}{\lambda \frac{\alpha}{2}} \qquad (72)$$

$$N_1 = \frac{\delta \sigma T_0^3}{\lambda_1} \qquad (73)$$

$$\bar{q}_{res}\left[\bar{t}(\bar{x})\right] = \bar{B}(\bar{x}) - \bar{H}_{z\to x}(\bar{x}) - \bar{H}_{y\to x}(\bar{x}) \qquad (64a)$$

$$\bar{B}(\bar{x}) = \varepsilon_1 \bar{t}^4(\bar{x}) + (1-\varepsilon_1)\left[\bar{H}_{y\to x}(\bar{x}) + \bar{H}_{z\to x}(\bar{x})\right] \qquad (65a)$$

$$\bar{H}_{y\to x}(\bar{x}) = \frac{1}{2}\int_0^{\bar{y}_1} \bar{B}_1(\bar{y}) \frac{\bar{x}\,\bar{y}\cos^2\frac{\chi}{2}\, d\bar{y}}{(\bar{x}^2+\bar{y}^2+2\bar{x}\bar{y}\sin\frac{\chi}{2})^{3/2}} \qquad (67a)$$

$$\bar{H}_{z\to x}(\bar{x}) = \frac{1}{2}\int_0^{1} \bar{B}(\bar{z}) \frac{(\bar{z}_0+\bar{z})(\bar{z}_0+\bar{x})\sin^2\gamma\, d\bar{z}}{\left[(\bar{z}_0+\bar{z})^2+(\bar{z}_0+\bar{x})^2-2(\bar{z}_0+\bar{z})(\bar{z}_0+\bar{x})\cos\gamma\right]^{3/2}} \qquad (66a)$$

$$\bar{z}_0 = \frac{z_0}{L_1}\ ;\quad \bar{y}_1 = \frac{y_1}{L_1} = 2\bar{z}_0 \sin\frac{\chi}{2} \qquad (74)$$

$$\bar{B}_1(\bar{y}) = \varepsilon_1 \bar{t}_0^4 + (1-\varepsilon_1)\left[\bar{H}_{x\to y}(\bar{y}) + \bar{H}_{z\to y}(\bar{y})\right] \qquad (68a)$$

$$\bar{H}_{x\to y}(\bar{y}) = \frac{1}{2}\int_0^{1} \bar{B}(\bar{x}) \frac{\bar{x}\,\bar{y}\cos^2\frac{\chi}{2}\, d\bar{x}}{(\bar{x}^2+\bar{y}^2+2\bar{x}\bar{y}\sin\frac{\chi}{2})^{3/2}} \qquad (69a)$$

$$\bar{H}_{z\to y}(\bar{y}) = \frac{1}{2}\int_0^{1} \bar{B}(\bar{z}) \frac{\bar{z}(\bar{y}_1-\bar{y})\cos^2\frac{\chi}{2}\, d\bar{z}}{\left[\bar{z}^2+(\bar{y}_1-\bar{y})^2+2\bar{z}(\bar{y}_1-\bar{y})\sin\frac{\chi}{2}\right]^{3/2}} \qquad (70a)$$

The boundary condition for equation (60b) is

$$\bar{T} = 1.0 \quad \text{at} \quad \bar{x} = 0 \qquad (62a)$$

The set of equations (60b), (61b), (64a) through (70a) in conjunction with the boundary condition (62a) determines completely the solution of the problem.

The solution of the written set of equations was developed by analogy with the solution of the problems considered in the previous sections of this report. The emission coefficient ζ was determined by the formula similar to formula (23). From the formula (23) and the above set of the equations it follows that for the rim of triangular profile ζ is a function of the parameters N, N_1, $\bar{e}_0$, ε_1 and γ.

For the case of the four-fin star-shaped radiator with the triangular fins the results of the numerical solution developed in the range of the emissivity $\varepsilon = 1,0 \div 0,75$ and of the values of parameter $N_1 = 0 \div 0,03$ are shown in figures 19 and 20. The emission coefficient relations $\zeta(\nu)$ at $N_1 = 0$ correspond to the case of the radiator without coating which has been considered in the first section of this paper.

In the process of the calculation the dimensionless thermal conduction parameter N has took on the values of 0, 0.25, 0,5, 1.0, 1.5, 2.0, 2.5, 3.0. The relations presented clearly illustrate the influence of the thermal resistance of coatings on the radiation efficiency.

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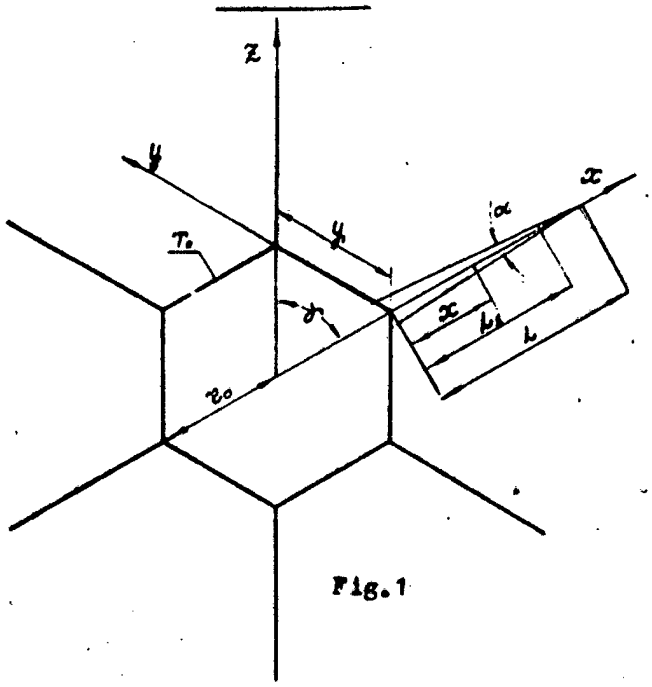

Fig.1

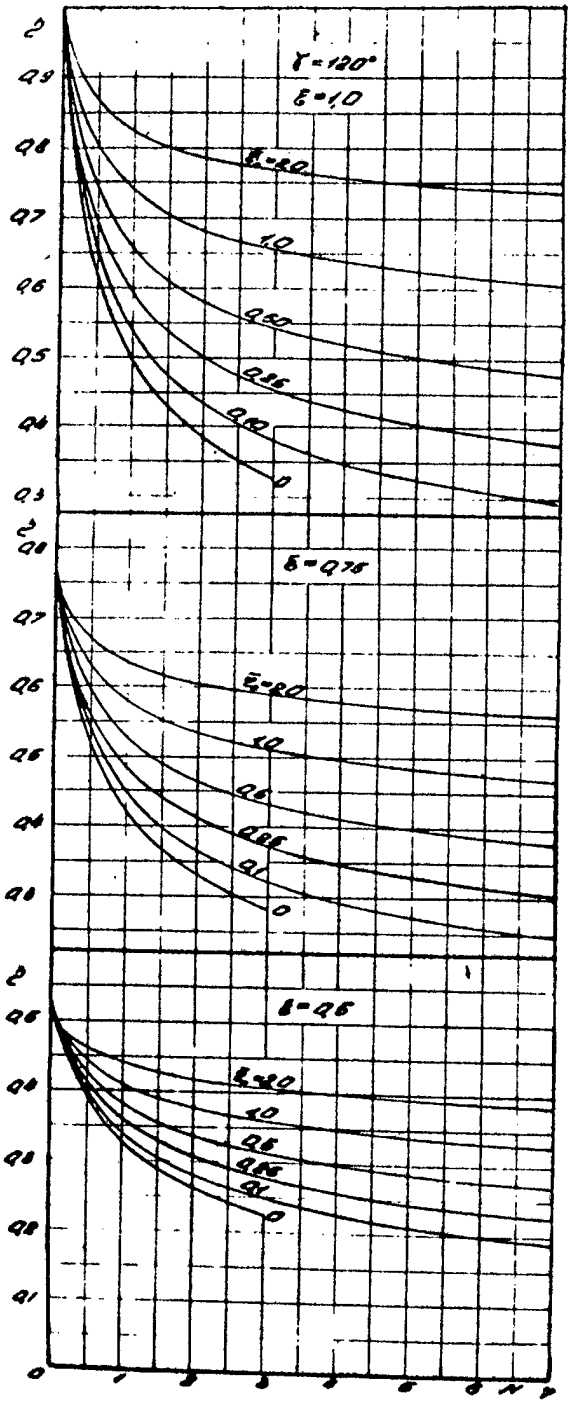

Fig.2

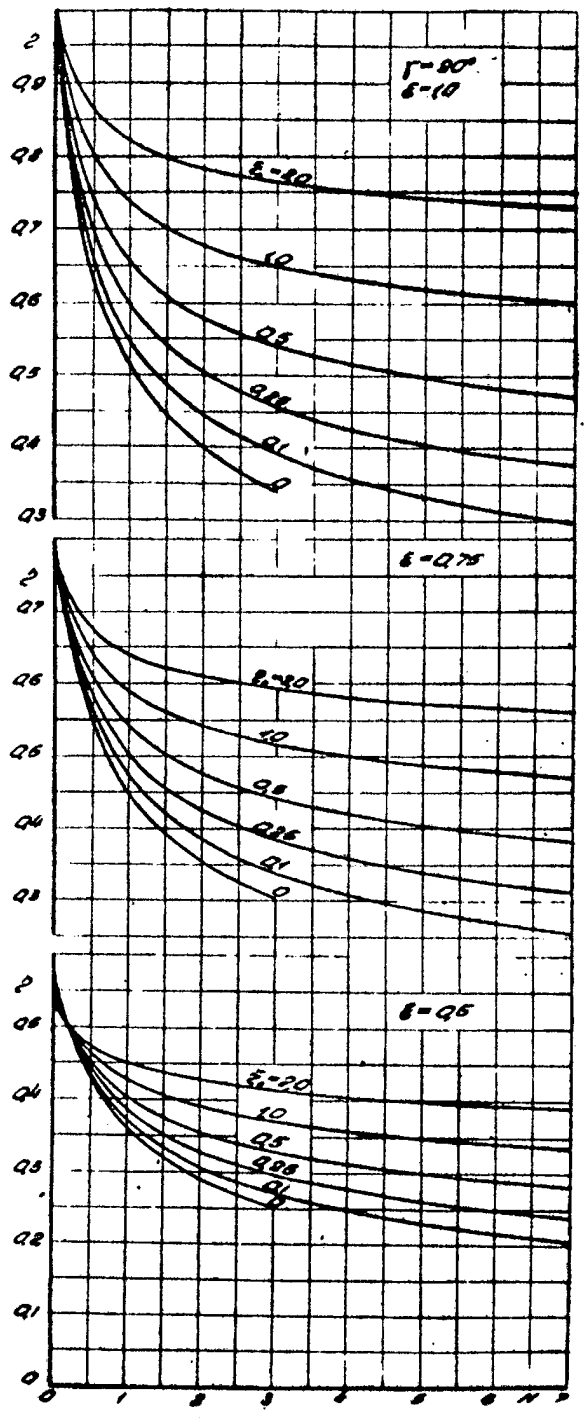

Fig.3

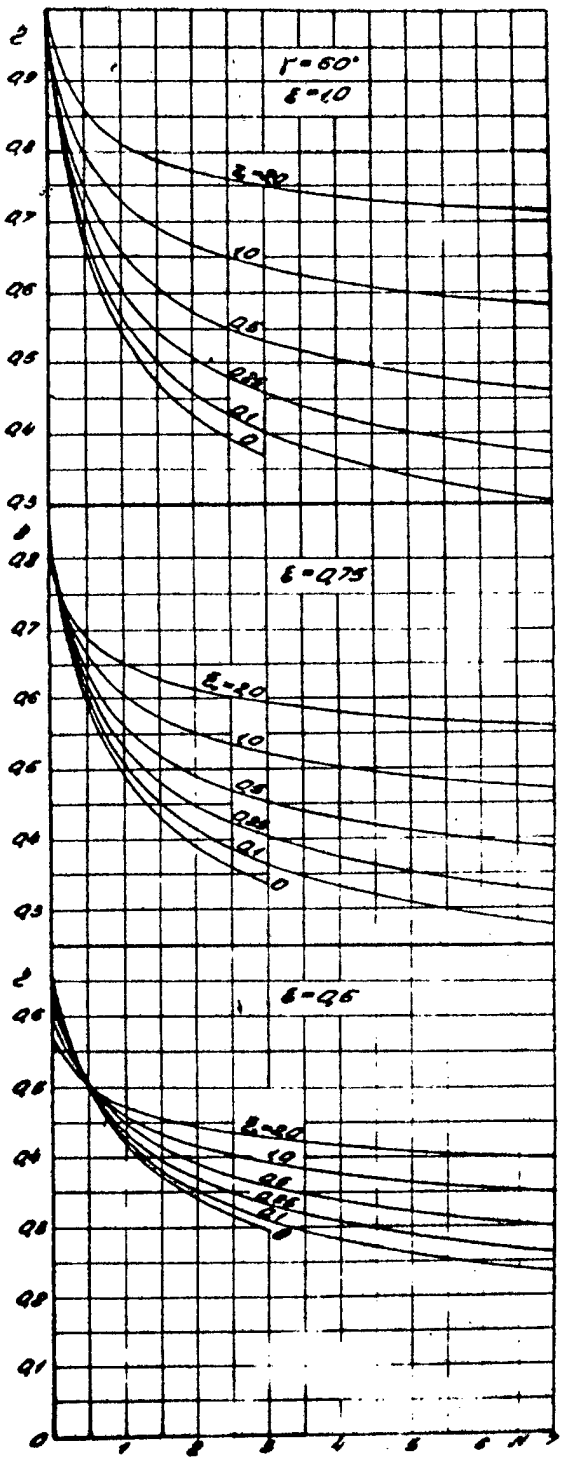

Fig.4

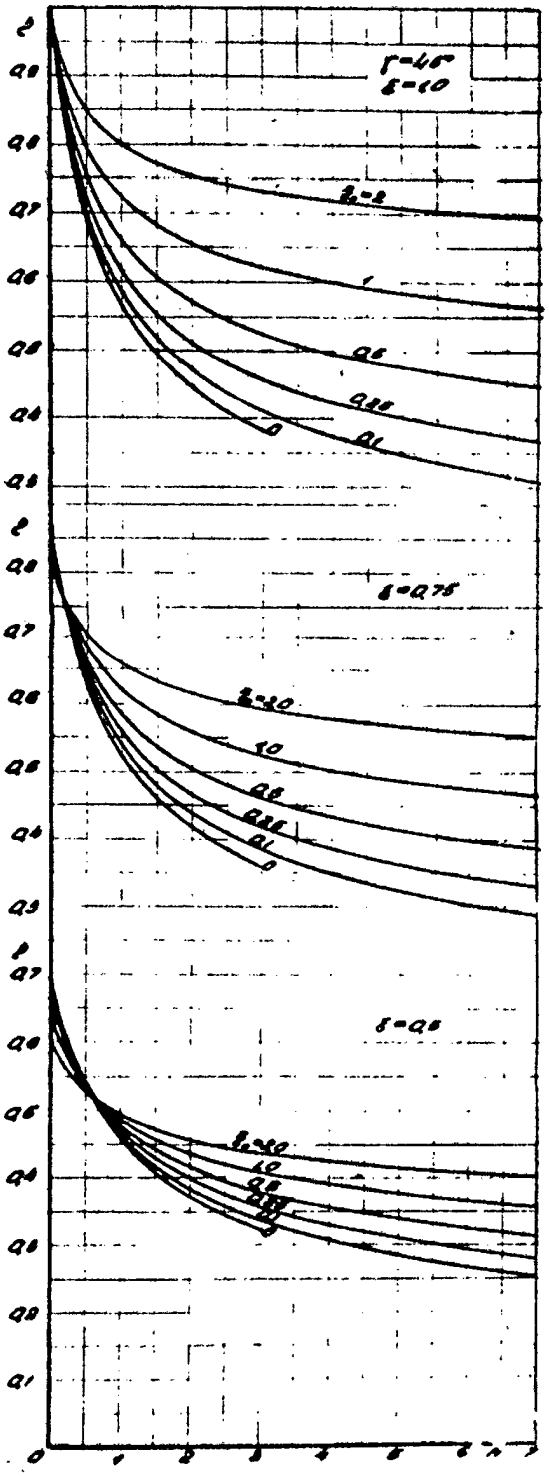

Fig.5

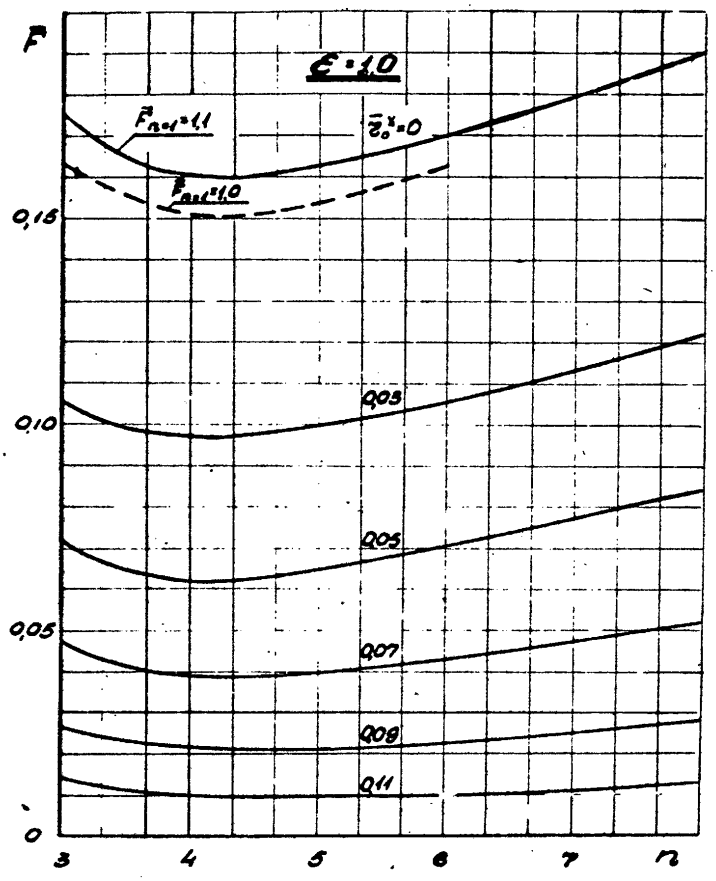

Fig.6

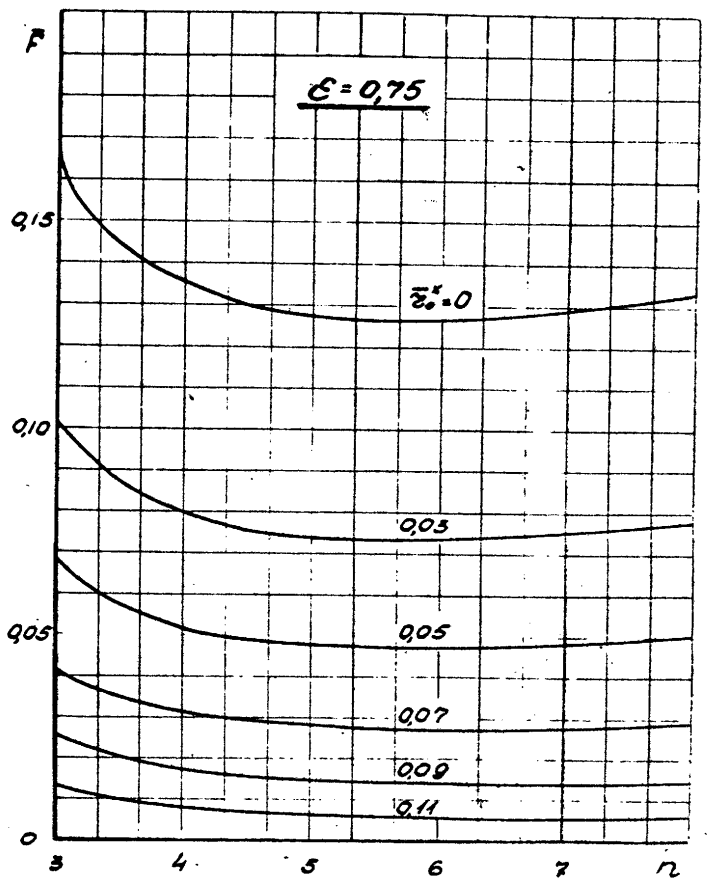

Fig.7

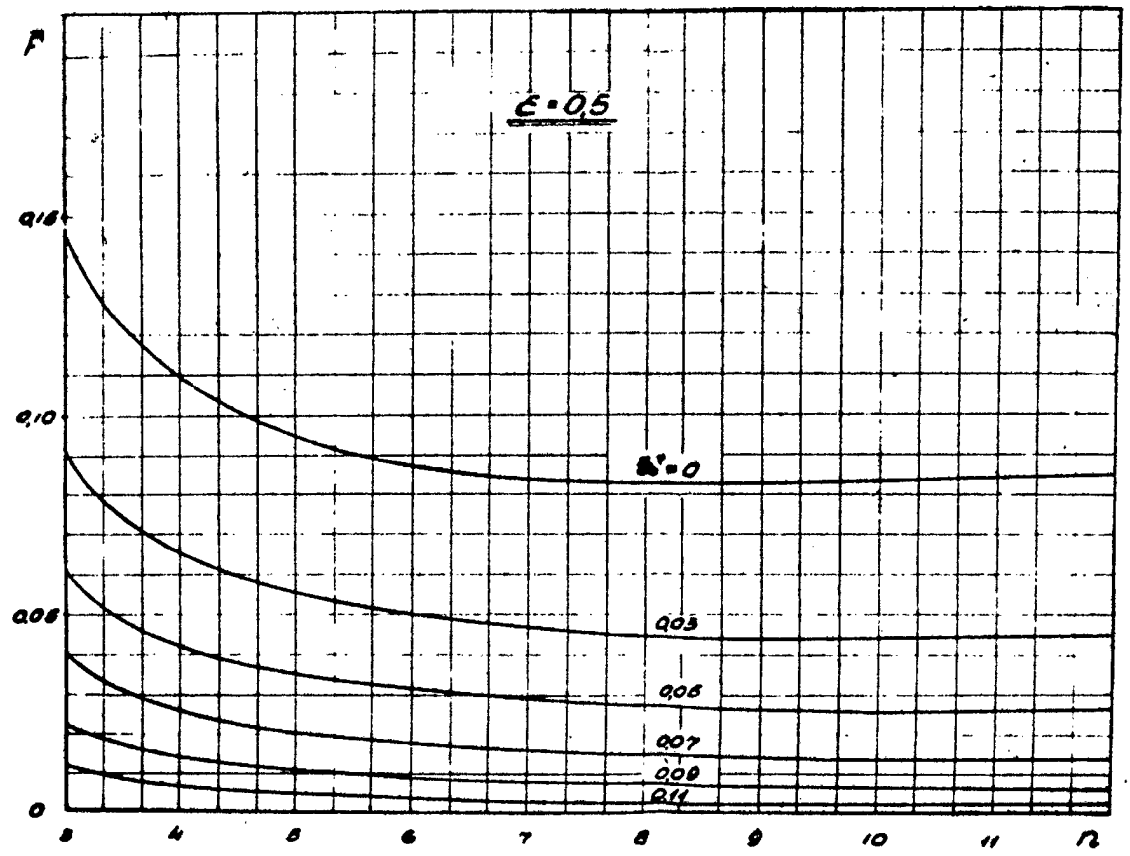

Fig.8

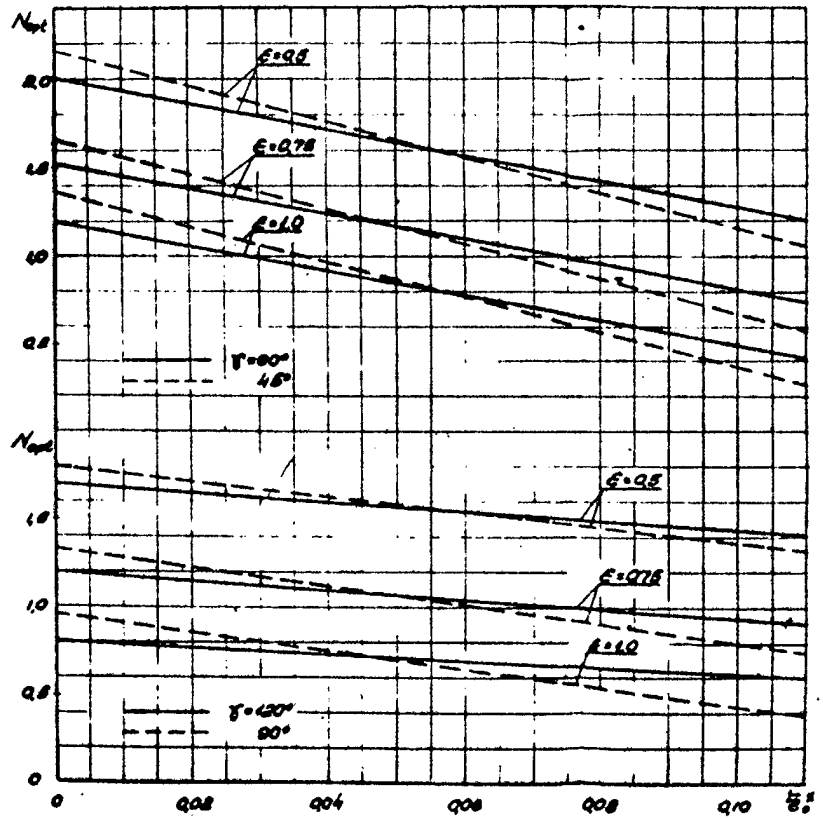

Fig.9

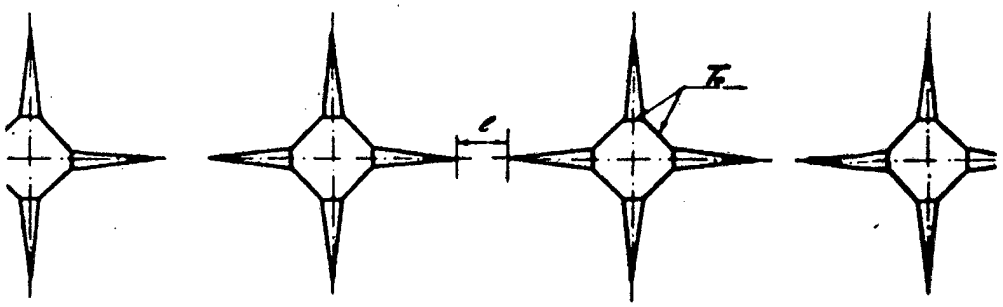

Fig. 10

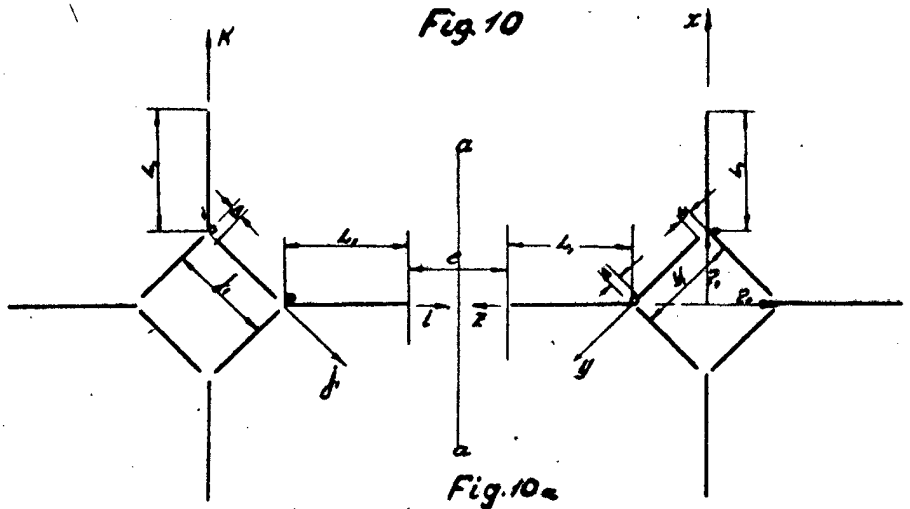

Fig. 10a

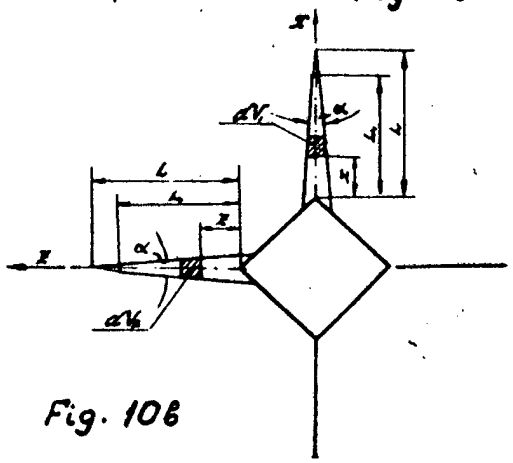

Fig. 10b

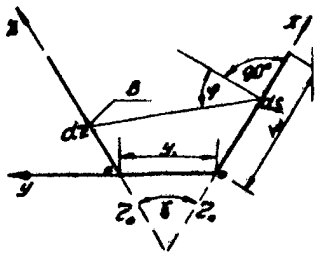

Fig. 10c

B-the effective radiation flow leaving the elemental area dz.

Fig. 10

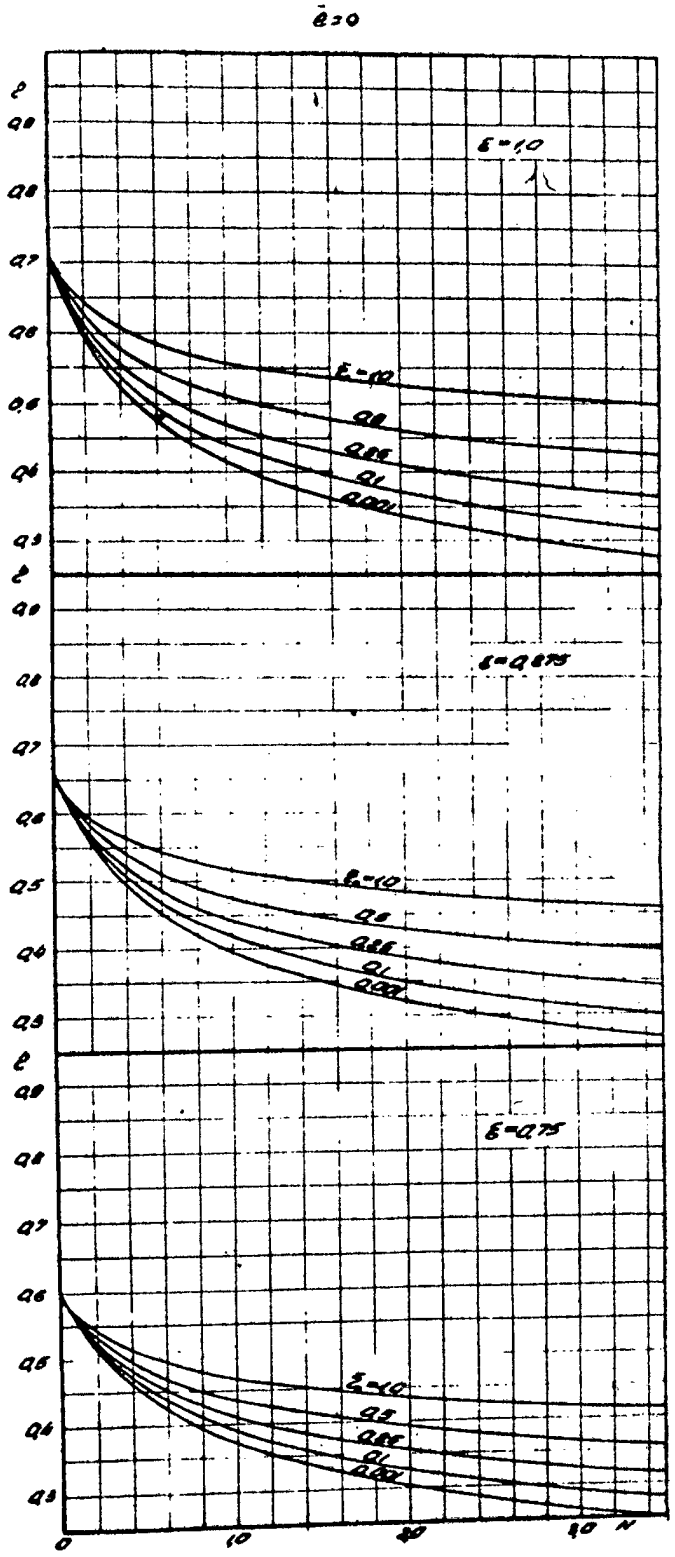

Fig. 11

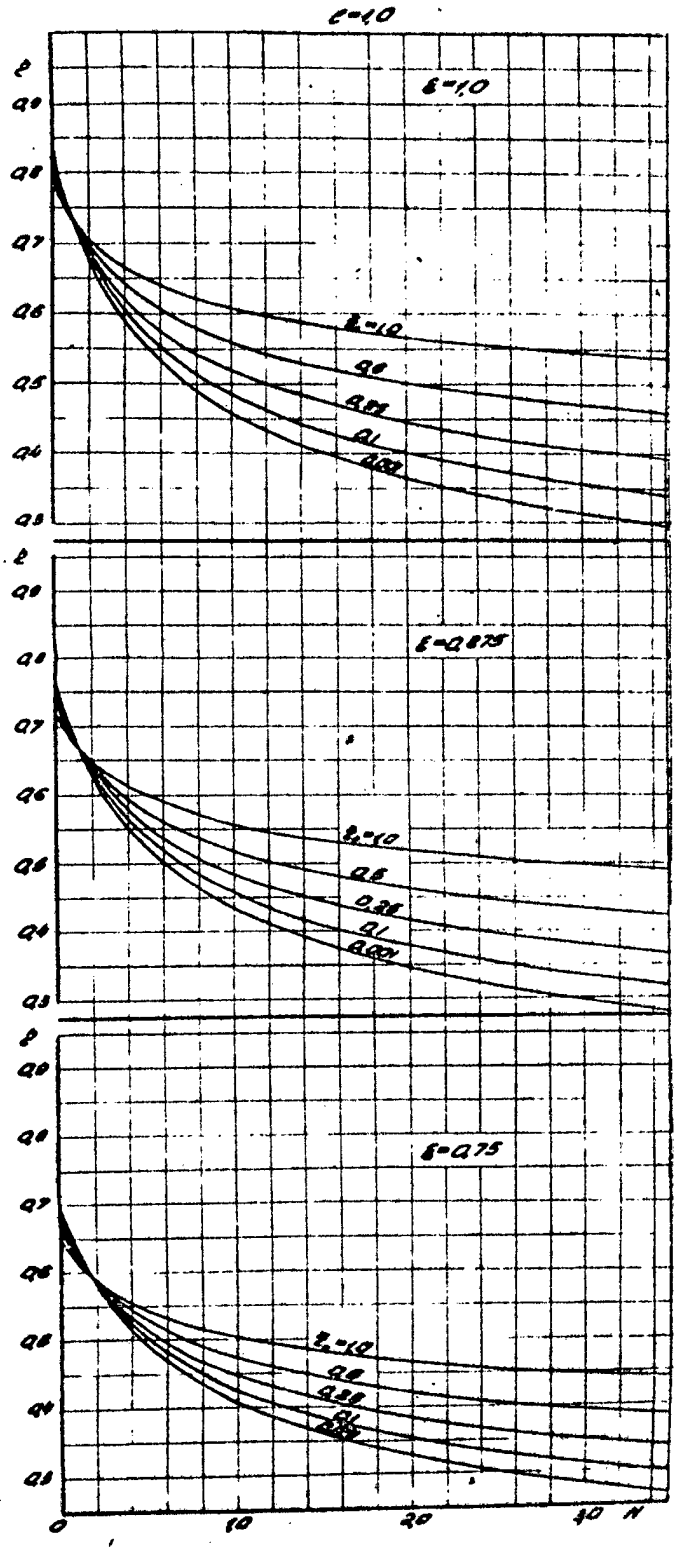

Fig. 12

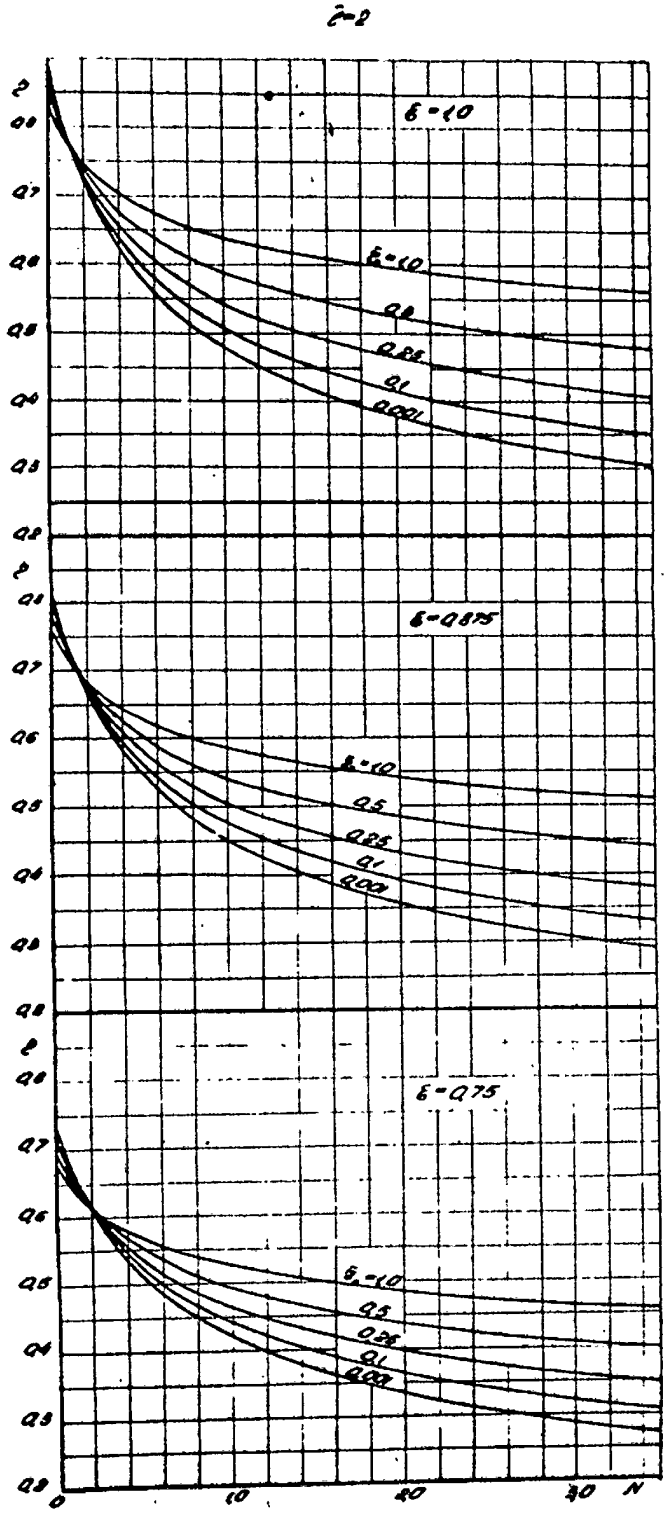

Fig. 13

Fig. 14

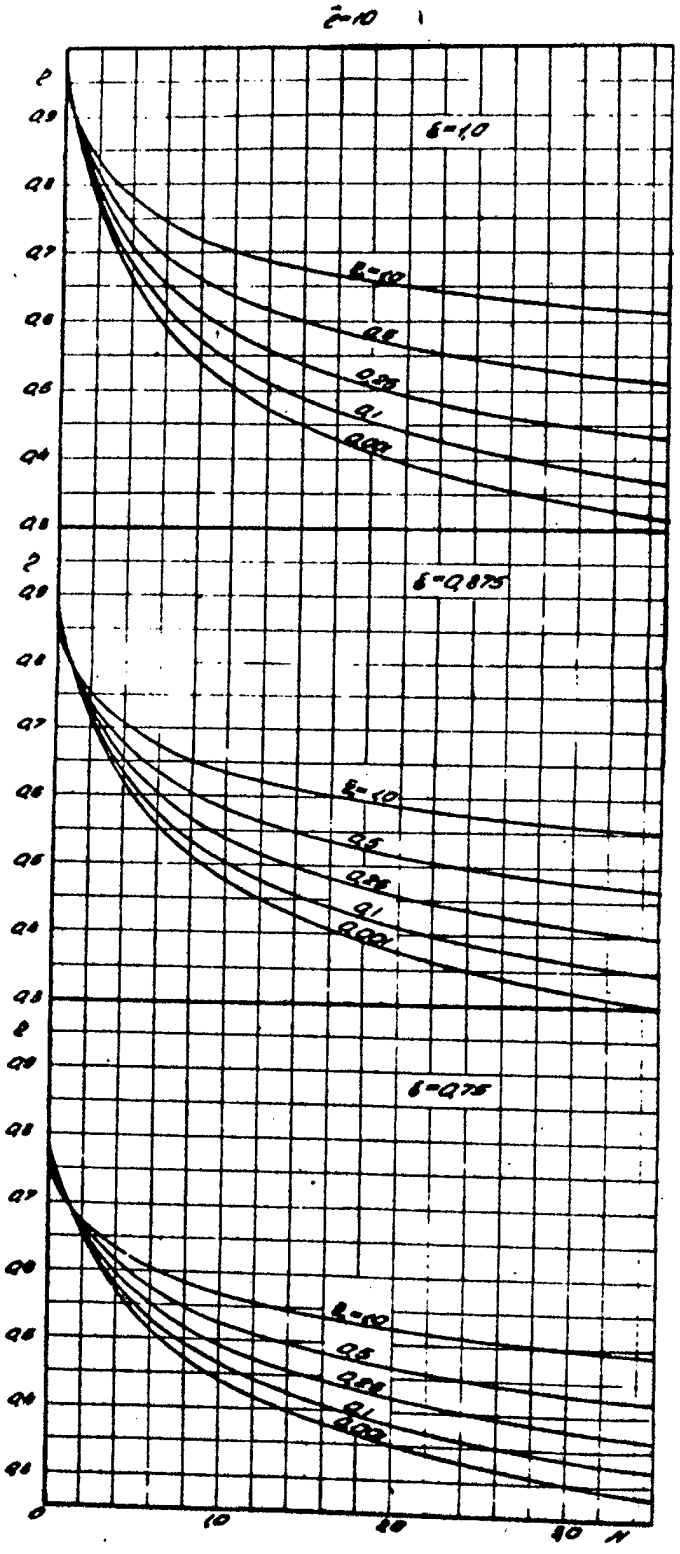

Fig.15

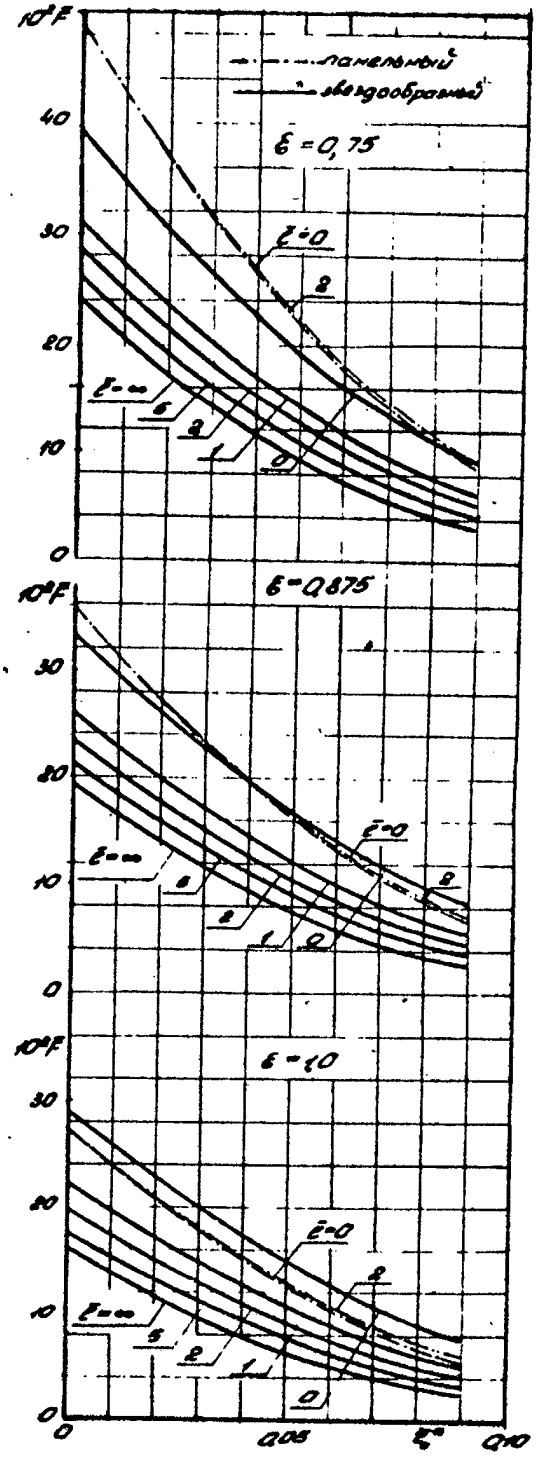

Fig.16

—·— pannel radiator
——— star-shaped radiator

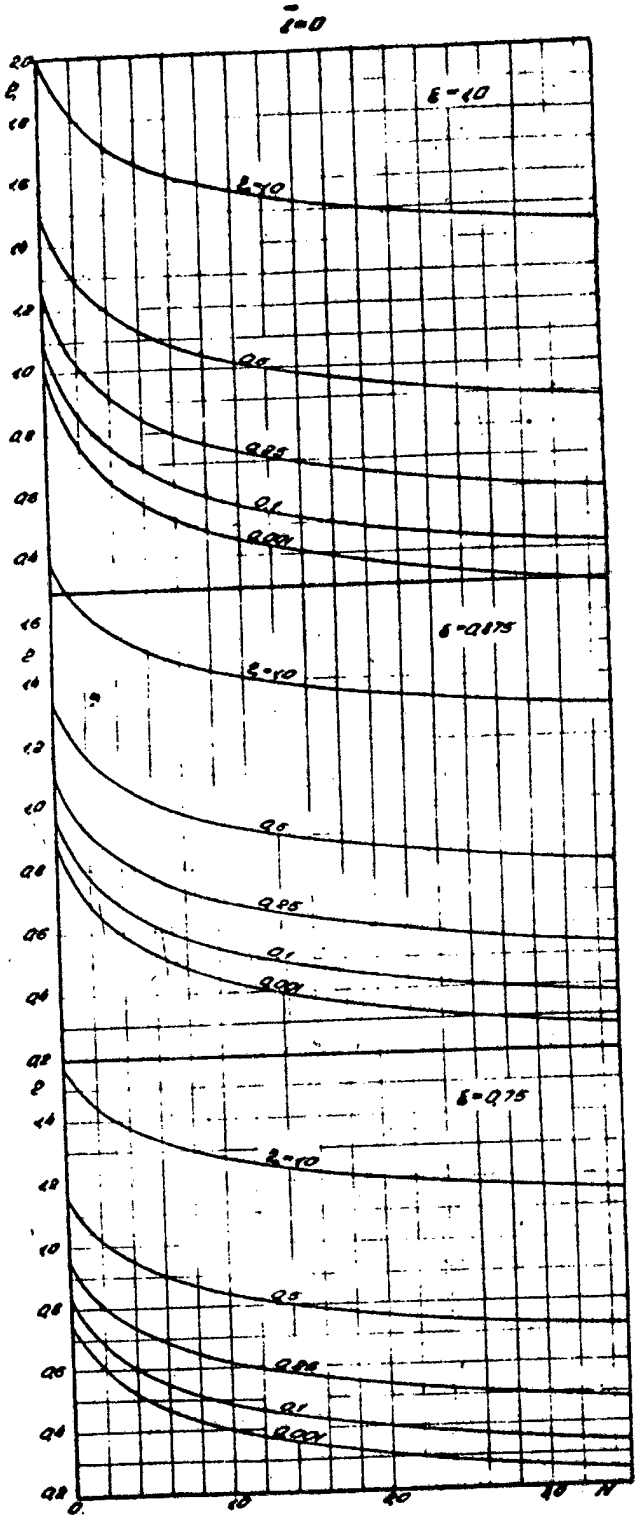

Fig. 17

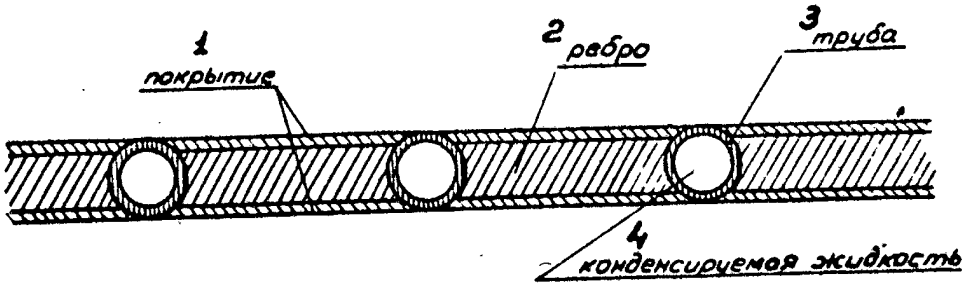

Fig. 18a

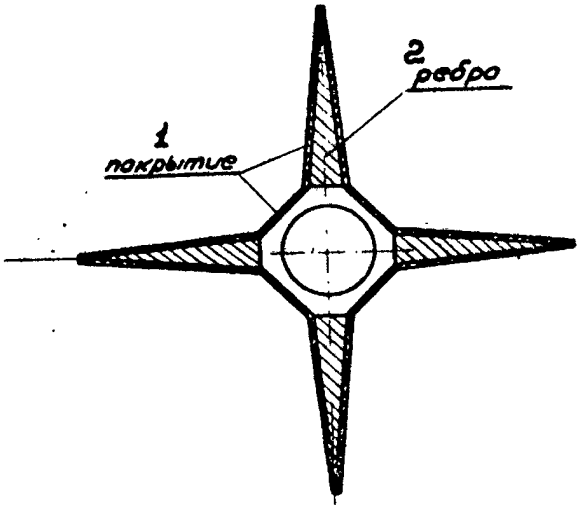

Fig. 18б

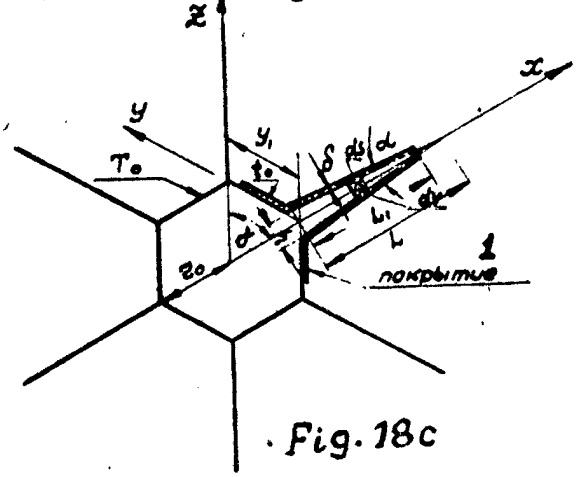

Fig. 18c

Fig. 18

1-coating
2-fin
3-tube
4-condensed fluid

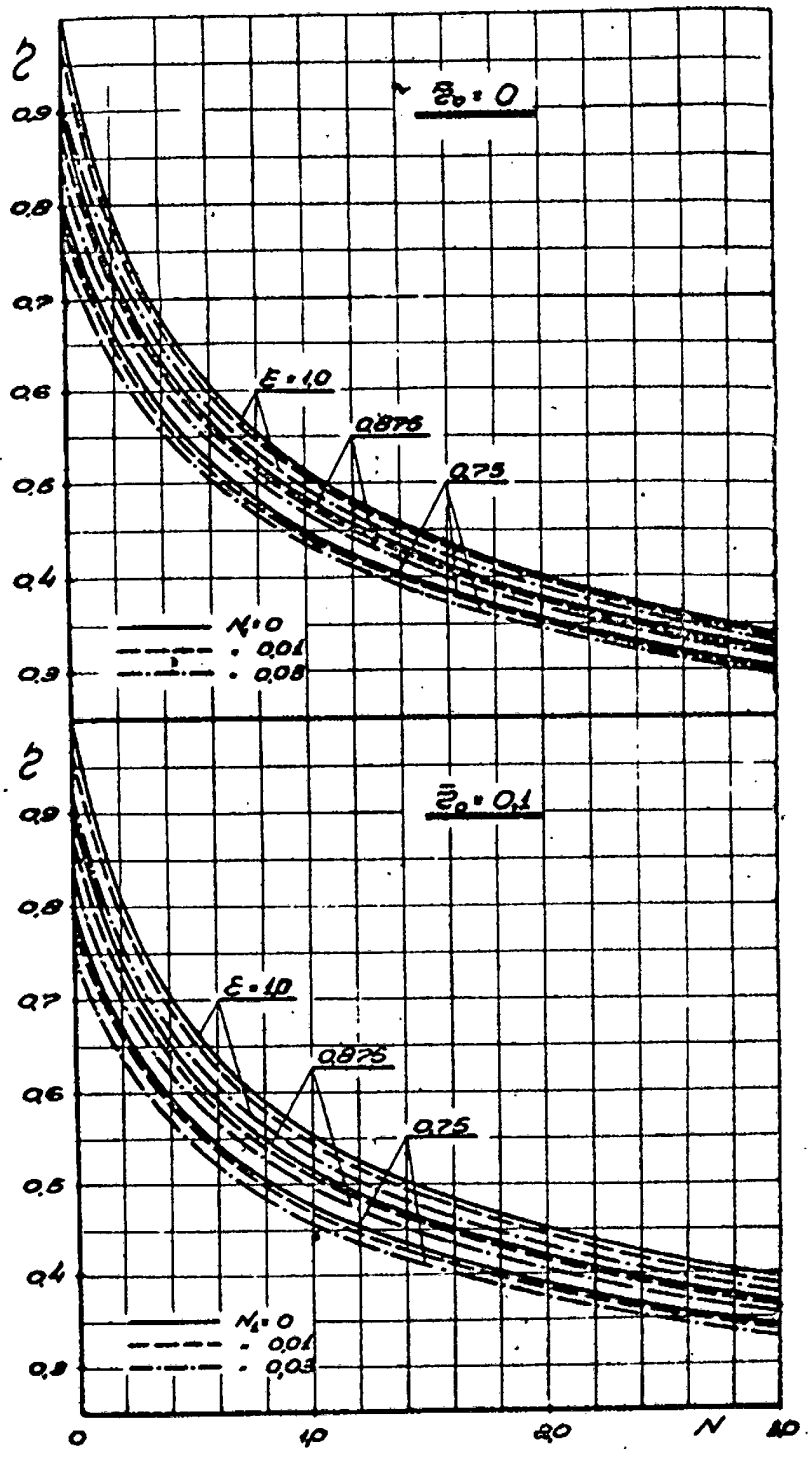

Fig. 19

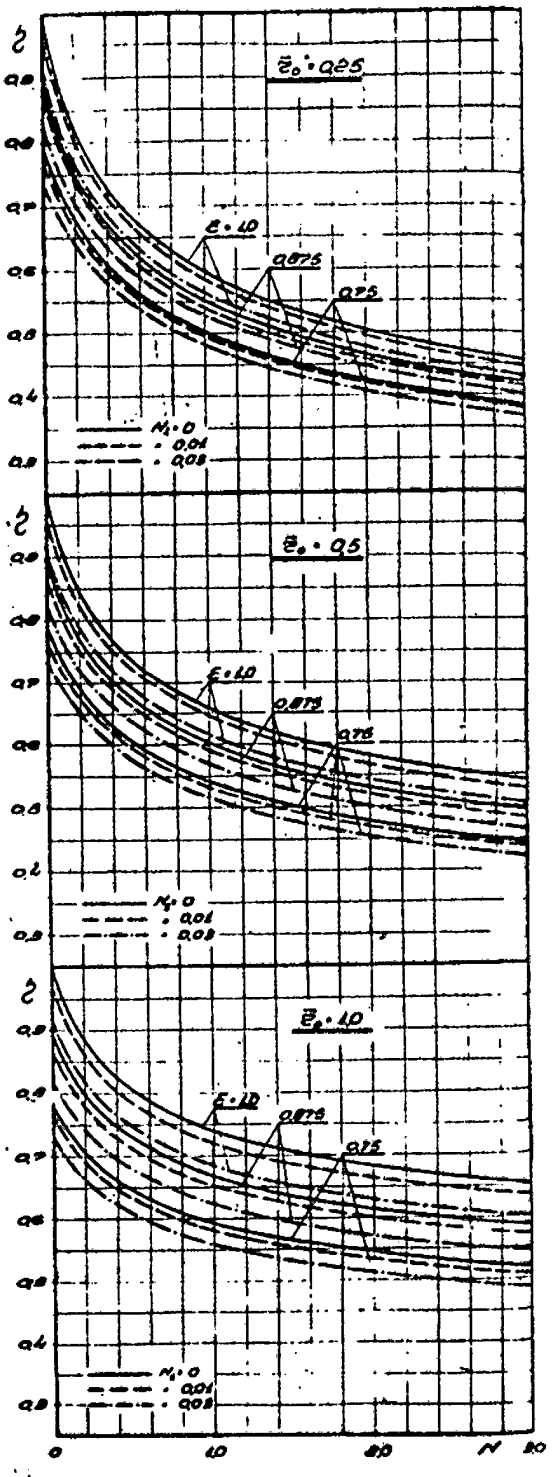

Fig. 20

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LOW THRUST SPACE FLIGHT MECHANICS

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S u m m a r y

The results of more than two hundred investigations are summarized in this paper which have laid the foundation of a new branch of mechanics - low thrust flight mechanics.

It includes the combined solution of the problems associated with selection of optimum weight parameters, the optimum propulsion system control parameters and determination of optimum trajectories for continuously applied propulsive thrust for a case when the weight of the propulsion system is comparable to the launching weight of a vehicle.

The combined solution of the three problems involved is caused by the peculiarities of low thrust propulsion systems: their continuous work and large weight.

The report includes the following sections:

Introduction. The physical principles of low thrust and general formulations of optimization problem in space flight mechanics.

Part I. Space flight mechanics by solar sail.

1. Basic relationships and formulation of the problem.
2. Interplanetary transfer.
3. Planetary escape.

Part II. Power-limited flight mechanics.

4. Optimum weight relationships and elementary movements
5. Ideally controllable propulsion system - optimum weight relationships and optimum power control.
6. Ideally controllable propulsion - system optimum trajectories and vector acceleration programs due to thrust.
7. Uncontrollable propulsion system - optimum weight relationship, optimum trajectories and thrust vector direction programs.
8. Optimum constant and limited thrust acceleration trajectories.
9. Propulsion system controllability limitations.
10. Time limitations for a propulsion system.
11. The number of regulator positions limitations.
12. Optimum combination of power-limited and limited exhaust velocity propulsion systems.
13. Reliability effect in the power-limited propulsion system optimization problem.
14. Correction effect in the power-limited propulsion system optimization problem.
15. Additional weight components for a power-limited propulsion system. Propulsion systems similar to power-limited engines.

Part III. Low thrust trajectory determination methods.

16. Analytical solution of dynamics equations.
17. Numerical methods of trajectory determination.

В докладе приведено систематизированное изложение результатов более двухсот исследований, заложивших основу нового раздела механики – механика космического полета с малой тягой.

Предметом этого раздела является совместное решение проблем выбора оптимальных весовых параметров, оптимального управления двигательной системой и определение оптимальных траекторий полета с продолжительно действующей двигательной установкой, вес которой соизмерим со стартовым весом аппарата. Необходимость совместного рассмотрения этих трех составных частей проблемы обусловлена отличительными особенностями двигателей малой тяги: длительным временем работы и большим весом двигательной установки. Доклад включает следующие разделы:

Введение. Физические принципы малой тяги и общая формулировка проблемы оптимизации в механике космического полета.

Часть I. Механика космического полета с солнечным парусом.

§ I. Основные соотношения и формулировка задачи.

§ 2. Перелет между орбитами планет.

§ 3. Выход из сферы притяжения планеты.

Часть II. Механика космического полета с двигателями ограниченной мощности.

§ 4. Оптимальные весовые соотношения для простейших движений.

§ 5. Идеально регулируемая двигательная система – оптимальные весовые соотношения, оптимальное управление мощностью.

§ 6. Идеально регулируемая двигательная система – оптимальные траектории и программы для вектора ускорения от тяги.

§ 7. Нерегулируемая двигательная система – оптимальные весовые соотношения, оптимальные траектории и программы направления вектора тяги.

§ 8. Оптимальные траектории с постоянным и ограниченным ускорением от тяги.

§ 9. Ограничения на регулировочную характеристику двигательной системы.

§ I0. Ограничения на время работы двигательной системы.

§ II. Ограничения на число положений регулятора.

§ I2. Оптимальное сочетание двигательных систем ограниченной мощности и ограниченной скорости истечения.

§ I3. Учет надежности в задачах оптимизации для двигательных систем ограниченной мощности.

§ I4. Учет коррекции в задачах оптимизации для двигательных систем ограниченной мощности.

§ I5. Учет коррекции в задачах оптимизации для двигательных систем ограниченной мощности.

§ I6. Дополнительные весовые компоненты двигателя ограниченной мощности. Двигательные системы, родственные двигателю ограниченной мощности.

Часть III. Методы построения траектории с малой тягой.

§ I7. Аналитические решения уравнений динамики.

§ I8. Численные методы построения траекторий.

This paper is devoted to the summarized representation of the results of more than two hundred investigations, which have laid the foungation of a new branch of mechanics - low thrust space flight mechanics, the subject of which is the combinined solution of the problems, associated with selection of optimum vehicle weight parameters, of optimum propulsion system control parameters and determination of optimum trajectories for continuously applied propulsive thrust for the case, when the propulsion system weight is comparable with a vehicle launch weight.

Up to present theoretic investigations of the space flight mechanics have been advanced so much that one may speak of certain results of the work done. However material on this subject has been published in the form of hundreds of papers, included in different scientific journals; the present report, and a more detailed survey by the authors /24/, are attempts to summarise the principal aspects of the space flight machanics. Naturally, the authors have not set a task before themselves to analyse thoroughly all known works in this field, although to give an extensive bibliography was considered to be reasonable. The main emphasis in the paper has been made on the problem of the space flight optimization.

New aspects of the flight mechanics are integrant of the advanced space propulsion systems development. So the established between the XIX-th and XX-th centuries idea of space flight jet propulsion application has encouraged the development of the space flight mechanics (Mesh'erskii, Tsiolkowskii, Zander, Goddard, Hohmann and others) the science, studying motion of space vehicles as mass - variable bodies to determine the maximum payload conditions. In the first stage of the space flight mechanics development the most detailed investigation has been made on the motion of the thermochemically propelled rocket, for which the low specific weight of the propulsion system (the ratio of the propulsion system weight to the maximum developed thrust of the order of the units of percents) (see the below table 1) is characteristic.

For such rockets the neglect of the propulsion system weight has been allowable as a first approximation. and the problem of the flight optimization has been reduced to seeking for the condition of the lowest propellant expenditure to accomplish a prescribed space maneuvers. A reasonable choice of the rocket staging, e.g. the account for the consequently rejected tanks, the weight of which is proportional to the fuel reserve, has not principally changed the cited formulation. On the base of Tsiolkovskii's formula the condition is reduced to minimization of the simple kinematic parameter, called characteristic velocity, which demands consideration of the flight dynamics only: determination of the optimal space trajectories with an indication therein of the optimal phases with respect to application and orientation of the thrust vector. These problems have been represented in detail in the famous basic papers and monographs on rocketdynamics.

The development of advanced space propulsion systems, for example, solar sail, electric jet engines (see below table I) distinguished by their high weight ratio and by great possibilities the exhaust velocity mass

consumption control and others arose new problems of flight mechanics. The prospect of using such propulsion systems is determined by low mass consumption per unit thrust of electric jet and thermonuclear engines (because of high jet velocity) or by the absence of mass consumption for the solar sail. In this connection a new branch of flight mechanics have been developed - the space flight mechanics, studying the combination of the optimal relation between the weight components of a rocket, including the weight of the propulsion system main elements, optimal control and regulation of the propulsion system, and optimal space flight trajectories. The cited basic problem formulation of the space flight mechanics is integrant with the space propulsion system characteristics. Some space propulsion system characteristics are given in the table 1 (ref.94).

To the exclusion of the thermochemical propulsion systems the remaining ones are characterized by a high weight ratio. That is why the need for the combined consideration of the three cited aspects of the problem is due to the distinguished features of the related propulsion systems: the long time of their operation and their large weight.

It should be noted that the low thrust of a number of the discussed in literature propulsion systems (which are origins of the particular investigation advance) is not an outstanding feature of their physical principles, but only provides the assesment of the current possible parameters. That is why the current popular term "low thrust" is highly relative.

The discussed investigations cover a wide range of the flight conditions, for which, besides of the optimal flight dynamics, the selection of the optimal weight parameters is essential. The principal weight parameters considered are: the payload weight $G_n(t)$, the propellant weight $G_m(t)$; the power source weight $G_N(t)$, accelerator weight $G_p(t)$ and others. Then the total weight will be:

$$G(t) = G_n + G_m(t) + G_N(t) + G_p(t) + \cdots \qquad (1)$$

Relations between the weight parameters and propulsion system characteristics and the equation of motion form the complete set of equations, describing the vehicle behaviour:

$$G_m = -gq\left(-q\frac{V^2}{2}\right)_{max} = N_0\eta(V);\ G_N = \alpha(N_0)N_0;\ G_p = \gamma_i(P_{max})P_{max};\ P = qV;$$
$$\ddot{\vec{r}} = \frac{P}{G}q\vec{i}(t) + \vec{R}(\vec{r}, t) \qquad (2)$$

where q is mass consumption, V is an exhaust velocity, η is power- to thrust conversion efficiency, α is a power source specific weight, $\vec{r}$ is the radions-vector, $\vec{R}$ is the vector of the acceleration due to gravitational forces, $\vec{i}$ is a unit vector of the thrust.

To Zander F.A. /85/ (1964) belongs the first serious investigation of the problem of flight with propulsion systems, the weights of which are comparable with a launch vehicle weight. By Zander has been investigated in detail the solar sail flight (fig.1). These investigations were extended in current papers only after 1958.

The solar sail vehicle thrust is produced by radiation pressure from the sun, which is $P_0 = 0{,}94 \cdot 10^{-6}\ \frac{kg}{m^2}$ at Earth's orbit R_0. The magnitude P and direction ($\vec{n}$, $\vec{p}$) depend on the sail setting ($\vec{n}$, $\vec{e}$) (see fig.2) and reflectance :

4 $P = \frac{1}{2} P_0 \left(\frac{R_0}{R}\right)^2 S \sqrt{1 + \varepsilon^2 + 2\varepsilon \cos(2\widehat{\vec{n}\vec{e}})} \cdot (\vec{n}\cdot\vec{e})$

$(\widehat{\vec{n}\vec{P}}) = \arcsin \frac{(1-\varepsilon)\sin(\widehat{\vec{n}\vec{e}})}{\sqrt{1+\varepsilon^2+2\varepsilon\cos(2\widehat{\vec{n}\vec{e}})}}$ (3)

$P_0 = 6{,}944 \cdot 10^{-6} \frac{кГ}{м^2}$; $R_0 = 0{,}1495 \cdot 10^{12}$ м

In the papers, concerning the solar sail, the idealized case E = 1 is considered:

for $\varepsilon = 1$ $\vec{P} = P_0 \left(\frac{R_0}{R}\right)^2 \cdot S(\vec{n}\cdot\vec{e})^2 \vec{n}$ (4)

Then for S = const. the weight ingredient of the problem is essentially eliminated and the motion is characterized by the single constant control parameter a_0 (dimension is acceleration); so the only control function is the sail setting angle.

$q = 0$; $G_m = 0$; $G_N = 0$; $G(t) = G_0 - G_n + g\rho S\delta$ for S = const.

$\vec{\tau} = a_0 \left(\frac{R_0}{R}\right)^2 (\vec{n}\cdot\vec{e})^2 \vec{R}$; $a_0 = \frac{P_0 S}{G_n + \rho S \delta g} \le \frac{P_0}{\rho\delta}$ (5)

$\frac{G_n}{G} = 1 - \frac{a_0 \rho \delta}{P_0}$; $\frac{S}{G_0} = \frac{a_0}{g P_0}$; $|\vec{n}(t)| \equiv 1$ $\left(-\frac{\pi}{2} \le \widehat{\vec{n}(t)\vec{e}} \le \frac{\pi}{2}\right)$

for $\rho \sim 1 g/cm^3$; $\delta \sim 2{,}5 \cdot 10^{-5}$ cm, $a_{0\infty} \sim 0{,}4$ cm/sec^2

In a general case $(\vec{r}, t) \neq 1$ there will be no separation of the problem into weight and trajectory domains. Solar - sail - powered interplanetary flight in a central gravitational field of the sun in the cited formulation is studied in references: [85], [133], [139], [163], [225], [226], [144], [207], [188], [189], [28] and others and the equations of planar motion are analysed (see fig.3)

$R = r$; $(\vec{n}\cdot\vec{e}) = \cos\vartheta$; $R = -g_0 \left(\frac{r_0}{r}\right)^2$; $\dot{r} = v_r$

$\dot{v}_r = \frac{v_\varphi^2}{r} + (a_0 \cos^2\vartheta - g_0)\left(\frac{r_0}{r}\right)^2$; $\dot{\varphi} = \frac{v_\varphi}{r}$; $\dot{v}_\varphi = -\frac{v_r v_\varphi}{r} + a_0 \cos^2\vartheta \sin\vartheta \left(\frac{r_0}{r}\right)^2$ (6)

at $\vartheta = 0$ the solution is the conical sections

$v_\varphi r = const$; $\frac{1}{2}(v_r^2 + v_\varphi^2) - (g_0 - a_0) r_0^2 / r = const$

$v_r(0) = 0$; $v_\varphi(0) = \sqrt{g_0 r_0}$; $\ell = \frac{a_0}{g_0 - a_0}$ (7)

at $\vartheta = \vartheta_* = const$, the solution is the logarithmic spiral

$r = r_0 \exp(\varphi \operatorname{tg}\Psi)$; $\frac{\sin\Psi\cos\Psi}{2 - \sin^2\Psi} = \frac{\sin\vartheta_* \cos^2\vartheta_*}{g_0/a_0 - \cos^3\vartheta_*}$; $\Psi = \frac{v_r}{v_\varphi} = const$ (8)

However these particular solutions do not safisfy the boundary conditions; for example, in the circular initial and terminal orbits

opt $\vartheta(t)$; $t = 0$; $r = r_0$; $v_{r_0} = 0$; $v_{\varphi_0} = \sqrt{g_0 r_0}$; $t = T$; $r = r_1$;

$v_{r_1} = 0$; $v_{\varphi_1} = \sqrt{g_0 \frac{r_1^2}{r_0}}$. (9)

Exact solutions of the variational problem (T_{min} at a given value of a_0) are obtained numerically /144/, /28/. They are compared with the results of the above cited particular solutions at a prescribed terminal orbital velocity $v_1 = v_{1\delta}$ and at the optimal value of the terminal velocity in the orbit $v_{1\,opt}$ (the problem of misdistance). Comparison shows the great effect of the boundary conditions on the transfer time and on the velocity. The problem of the solar-sail vehicle escape from the planetary gravity field has been studied: (1) in an ecliptical plane, when the speed of the sail retation is half as high as the rotational speed of the vehicle about the planet, /202/ (see fig 6 and the equation of motion)

$\ddot{r} - r\dot{\varphi}^2 + g_0 \left(\frac{r_0}{r}\right)^2 = -a_0 \left(\frac{R_0}{R_p}\right)^2 \left|\sin\left(\frac{1}{2}\varphi\right)\right| \sin\varphi$

$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = a_0 \left(\frac{R_0}{R_p}\right)^2 \left|\sin\left(\frac{1}{2}\varphi\right)\right|^3$ (10)

and (2) in the orbit, perpendicular to the ecliptic plane, when the sail

setting is locally optimal with respect to the maximum rate of power increament /128/ (fig.7)

$$\ddot{x}^{(i)} + a_0\left(\frac{R_0}{R_p}\right)^2 \cos\alpha^{(i)}\cos^2\alpha^{(i)} + \frac{g_0 t_0^2 x^{(i)}}{\left[\sum_{i=1}^{3}(x^{(i)})^2\right]^{3/2}} \qquad i = 1,2,3 \qquad (11)$$

$$\cos\alpha^{(i)} \to \max(\vec{n}\cdot\hat{\vec{V}})$$

The second case (see the dotted line in fig.8) garantees a lettle shorter time to escape from the gravitational field, however, in both cases the nonoptimal programs of the sail setting have been studied.

From the table II /28/, /202/ it follows that the time to escape from the planetary gravitational field is a considerable part of the interplanetary flight time. In solar sail flight mechanics as a whole only partial conditions of the optimal transfers are investigated.

The other type of the propulsion system, the weight of which is comparable with a launch weight, is a power - limited electrical jet propulsion system. The principal schemes of such propulsion systems are presented in fig.9 taken from Preston - Thomas' work /190/ (1952). The systems consist of the power source No and the accelerator, converting the source energy into kinetic energy of the jet flow at the velocity V and the thrust P:

$$P = qV;\ \frac{1}{2}qV^2 = N \le N_{max} \le N_0\eta(V);\ G_N, G_p, G_m, G_\varepsilon. \qquad (12)$$

The primary energy source can be the Sun or the nuclear system (fig 9). In the range of V = 10-100 km/sec for nuclear systems the propelant consumption G is unessential /204/

$$\frac{1}{2}G_m V^2 = \varkappa c^2 G_\varepsilon;\ \varkappa \approx 5\cdot 10^{-4}: \qquad \begin{array}{lccc} V\ \frac{KM}{CEK} & 10^3 & 10^2 & 10^1 \\ G_\varepsilon/G_m & 10^{-2} & 10^{-4} & 10^{-6} \end{array} \qquad (13)$$

That is why the weight principal components are G_{N1} G_p and G_m. The source G_N of limited power N_0 determines the main properties and the name of this kind of the propulsion systems.

The specific element of the power source is a radiator for the radiation heat output in a thermoenergetic cycle. The investigations in /16/, /19/, /20/, /22/, /74/, /82/, /97/, /100/, /206/, /228/, /230/ and others have shown the optimal geometry of the heat radiating elements, which provide the minimum weight at the prescribed heat flow Q_0 and the temperature T_0; the examples of the optimal shaping of the heat radiating fins are shown in fig (10), (16), (19). Note that the weight of the optimal heat radiating elements is proportional to $\sim -\frac{Q_0^2}{T_0^3}$. Together with the determination of the power unit weight, the total weight of the power source, which depends on the value of the maximal power $G_N = \alpha(N_0)\,N_0$, is determined (see fig.11) /107/.

The characteristics of electrical acceleration, transforming the source energy into the kinetic energy of the oriented jet flow, are considered in detail in a number of works /42/, /49/, /75/, /214/, /220/, and others. In the arc accelerators (fig.12) thermal energy of the propellant heating up to the temperature T_0 is used. The exhaust velocity control may be

accomplished by the input power N and by the consumption q :

$$V = A\lambda c\sqrt{\frac{2\varkappa}{\varkappa+1}gRT_0} \qquad T_0 = f\left(\frac{N}{q}\right) \tag{14}$$

Accelerator efficiency η depends essentially on the exhaust velocity (fig.13); the exhaust velocity is bounded by the thermal regimes.

Accordingly, for electrostatic (ion) accelerators (see fig.14) the efficiency η depends essentially on V, and increase with the exhaust velocity

$$\eta = \frac{V^2}{V^2+U} \qquad U \sim q$$

Here again the magnitude of the exhaust velocity V and of the thrust P can be controlled by varying the electrical parameters.

In general the accelerators permit the power to be converted into thrust with the efficiency, dependent on the exhaust velocity in some velocity range. The mentioned power source and accelerator properties detemine the propulsion system characteristics, needed for the flight mechanics analysis, and the thrust control capabilities and weight consumption for thrust realization.

Between the principal parameters of the power - limited propulsion system there are two simple relationships: $q, V, P, N: \quad P = qV$

$$N = \frac{1}{2}qV^2 \le N_{max} \le N_0(t)\eta(V) \tag{15}$$

therefore the independent parameters are those two, for which it is reasonable to take the explicitly constructed ones. Below for the three variants there given full set of equations.

$$V \le V_{max},\ (N,V): \quad P = \frac{2N}{V};\quad q = \frac{2N}{V^2}$$

$$q \le q_{max},\ (N,q): \quad P = \sqrt{2Nq};\quad q$$

$$0 \le q \le \infty,\ 0 \le V \le \infty: \quad P = a\frac{G}{g};\quad q = \frac{a^2G}{2Ng^2}$$

$$G(t) = G_n + G_N + G_m + G_p;\quad G_N = \alpha(N_0)N_0(t);\quad G_p = \gamma(P_{max})P_{max}, \tag{16}$$

$$G_m = -gq; \qquad \dot{G}_m = -g\frac{2N}{V^2} \qquad \ddot{\vec{r}} = \frac{2N_0}{VG(t)}\vec{i} + \vec{R}$$

$$\dot{G}_m = -gq \qquad \ddot{\vec{r}} = \frac{\sqrt{2Ng}}{G(t)}\vec{i} + \vec{R}$$

$$\dot{G}_m = -\frac{a^2G^2(t)}{2N_0} \qquad \ddot{\vec{r}} = a\vec{i} + \vec{R}$$

$\vec{i}(t)$, $N(t)$, $G_N(t)$, $G_p(t)$ and $a(t)$, or $V(t)$, or $q(t)$.

The variational problem requires the determination of the optimal control of the functions, presented in the last line of (16), and furnishing the maximum payload $\bar{G}_{n\,max}$ at a specified maneuver and at a specified maneuver time T. Here in contrast to the classical astrodynamic problems, a weight problem with respect all components must be optimally solved in combination with the dynamic problem .

The weight component role may be illustrated by a number of the simulation problems [5], [8], [11], [14], [54], [75], [90-95], [214], [220], and others. The obvious problem is that of optimal motion with

constant thrust P in a specified interval of time T /115/:

$$N = const;\ G_N = \alpha N = const;\ G_p = 0;\ P(t) = const$$

$$P, T;\ G_n = G_0 - (G_m + G_N) = G_0 - P\left(\frac{2}{V}T + \frac{\alpha}{2}V\right)$$

$$\max G_n \rightsquigarrow V_{opt} = \sqrt{\frac{2gT}{\alpha}};\quad G_m = G_N = P\sqrt{\frac{1}{2}g\alpha T} \qquad (17)$$

for which the maximum G_n is reached at $G_m = G_N$ (see the curve "3" in fig.15). With this problem as an example it is shown [18] that at the related level of the parameters $\frac{T}{\alpha} \sim 1\ \frac{year \cdot kwt}{kg}$ the influence of the relativistic corrections is unessential

$$G_n = G_0 - P\left(\frac{2T}{V}\sqrt{1-\frac{V^2}{c^2}} + \alpha V\,\frac{1-\sqrt{1-V^2/c^2}}{V^2/c^2}\right)$$

$$V_{opt} \approx \sqrt{2\frac{gT}{\alpha}}\left(1-\frac{5}{4}\frac{gT}{\alpha c^2}\right) = \frac{\sqrt{2\frac{gT}{\alpha}}}{1-\Delta},$$

for $\frac{T}{\alpha} \sim 1\ \frac{year \cdot kwt}{kg}$ $\quad \Delta \sim 10^{-6}$ $\qquad (18)$

The examples of the optimal weight relationships ("1" - P = const, max velocity increment ΔV; "2" - P = const, travel along the distance L at min T; 4 - acceleration G = const, L at min T, 5 - a = const, max ΔV) show the essential influence of the problem type on the solution. The values of max ΔV and min T for the mentioned simulation problem are given in fig.16. Analysis of the problems, from which power-limited motion investigation started, has shown one feature of the considered optimization - the need for selection of the optimal relation between the weights of the power source and the propellant. The further progess of the simulation problem is the investigation of the optimal flight with an ideally controlled propulsion system [139] [140], [15] and others. When there is no restrictions:

$$0 \le P \le \infty;\ 0 \le q \le \infty;\ 0 \le V \le \infty;\ N(t) \le N_{max} = N_0 ?;\ \eta = const$$

$$G_p = 0;\ G_N(t) \equiv G_{N_0};\ G(t) = \frac{G_0}{1 + \frac{G_0}{2g}\int_0^T \frac{a^2}{N}dt}$$

$$\bar{G}_n = \frac{G_n}{G_0} = \frac{1}{1 + \frac{G_0}{2g}\int_0^T \frac{a^2}{N}dt} - \bar{G}_N;\quad opt\,N(t) = N_{max} \qquad J = \int_0^T a^2 dt \qquad (19)$$

при $G_N = \alpha N_0 (\alpha = const)$: $\bar{G}_n = \frac{1}{1 + \Phi/\bar{G}_N} - \bar{G}_N$; $\quad \Phi = \frac{\alpha}{2g\eta}J$

$$\bar{G}_{n_{max}} = (1-\sqrt{\Phi})^2;\ \bar{G}_{N opt} = \sqrt{\Phi} - \Phi$$

the maximum payload $\bar{G}_{n\,max}$ is reached as the functional is minimal. Separation of the total problem into a weight and a trajectory ingredients is essential, and justifies the idealization. The mentioned separation retains with the discussed above (see fig.11) nonlinear relationship between the source weight and the maximum power /21/, /24/:

$$\text{at } G_N = \alpha(N_0)N_0 \quad opt \rightsquigarrow \frac{\alpha G_N(N_0)}{\alpha N_0}\left(\frac{N_0}{G_N}\frac{1}{\sqrt{J/2g\eta}} + \frac{1}{\sqrt{J/2g\eta}}\right) = 1 \qquad (20)$$

For power approximation $\alpha(N_0)$:

$$\text{at } G_N = A N_0^{\chi};\quad \bar{G}_{n_{max}} = 1 + \Phi - \sqrt{\Phi}\left(\sqrt{\chi} + \frac{1}{\sqrt{\chi}}\right);\quad \Phi = \frac{G_N}{2gN_0}\int_0^T a^2 dt \qquad (21)$$

For the thermojet propulsion systems, due to their low specific weight, the problem of the optimal staging is in principal the problem of the optimal tank rejection. For the power-limited propulsion systems, due to high power source, relative weight, the problem of the optimal staging arises in a new formulation, namely the choice of the optimal power source weight decrease, accompanied by proportional maximum power decrease. As is shown in our work [15] for the general case and later in [166] for the particular case α = const, at a multistaging power-source elements rejection for the idealized system the weight and trajectory problem ingredients are separated: $\frac{\alpha G_N(t)}{dt} \le 0$; $N_0(t) = \frac{G_N(t)}{\alpha}$; $N(t) = N_0(t)$

$$G_{n_{i-1}} = G_{n_i} + \Delta G_{m_i};\quad \Phi_i = \frac{\alpha}{2g}\int_{t_{i-1}}^{t_i} a^2 dt;\quad opt\ \frac{G_{n_i}}{G_{n_{i-1}}} = \left(\frac{1-\Phi_i}{1+\Phi_i}\right)^2 \qquad (22)$$

$$\bar{G}_{n_{max}} = \left(1-\sqrt{\Phi_1}\right)^2 \sum_{i=2}^{n}\left(\frac{1-\Phi_i}{1+\Phi_i}\right)^2;\qquad \Phi = \sum_{i=1}^{n}\Phi_i$$

The limited case is that of the infinite number of elements /29/:

$$opt.\ G_N(t):\ \bar{N} = \frac{N}{N_0};\quad 1 \ge \bar{N}(t) \ge 0$$

$$\dot{G}_m = -\frac{G^2(t)}{G_N \bar{N}}\frac{\alpha}{2g}a^2;\qquad \dot{\vec{r}} = \vec{v};\quad \dot{\vec{v}} = a\vec{\tau} + \vec{R}$$

$$T_{min},\ \bar{G}_n;\quad (\vec{r}_0, \vec{V}_0);\ (\vec{r}_1, \vec{v}_1)$$

$$t=0:\ \bar{G}(0) = \bar{G}_n + \bar{G}_{N_0} + \bar{G}_{m_0} = 1;\quad t_1 = T;\quad \bar{G}_{m_1} = 0 \qquad (23)$$

$$opt\quad \bar{G}_N(t) = \begin{cases} \bar{G}_m + \bar{G}_n \\ const \end{cases} \quad u\ min\ \Phi = \frac{\alpha}{2g}\int_0^T a^2 dt$$

$$\bar{G}_N(t) = const \qquad при\ 1 \ge \bar{G}_n \ge 0{,}25$$

Data in fig 18 show, that with optimal power-source weight decrease the payload $\bar{G}_u$ slightly increases at high values Φ.

If a part γ of the weight of the rejected power-source elements is used according to Zander's ideas, as a propellant /79/:

$$q_\Sigma = q + \gamma\frac{\alpha G_N}{dt};\quad \gamma(t) \le \gamma_{max} \le 1;\quad \dot{\bar{G}}_m = -\bar{q};\quad \dot{\bar{G}}_N = -q_N;\quad \dot{\vec{r}} = \vec{v}$$

$$\dot{\vec{v}} = \frac{\sqrt{2k\bar{G}_N\bar{N}(\bar{q}+\gamma\bar{q}_N)}}{\bar{G}_m + \bar{G}_N + \bar{G}_n}\vec{\tau} + \vec{R};\qquad \bar{q} = \frac{qV}{G_0};\quad opt\ \gamma(t) = \gamma_{max} \qquad (24)$$

then the payload can be sufficiently increased (see fig.19). And in this case trajectory and weight ingredients of the problem are separated, which is the main property of the ideally controlled limit thrust propulsion

system.

The solution of the trajectory ingredient of these problems must give the minimum of the functional $\int_0^T a^2 dt$ along the trajectory. Such an optimal control has been investigated in [15], [139], [140], [3], [34] and others. The optimal trajectories and the laws for the acceleration-due-to-thrust vector for a planar motion in the field of the two gravitational centers are determined by the solution of the following set of the differential equations [15] (see fig. 20).

$$\text{Opt } \vec{r}(t),\ \vec{a}(t):\ \ddot{\vec{r}} = \vec{a} + \vec{R}(\vec{r}, t);\ \left(\vec{r}(0)=\vec{r}_0;\ \dot{\vec{r}}(0)=\dot{\vec{r}}_0\right);\ \left(\vec{r}(T)=\vec{r}_T;\ \dot{\vec{r}}(T)=\dot{\vec{r}}_T\right);\ \min J = \int_0^T a^2 dt$$

$$\ddot{r}^{(i)} = a_r^{(i)} + r^{(i)}\left[\dot{\varphi}^{(i)}\right]^2 - \frac{K^{(i)}}{\left[r^{(i)}\right]^2} - R^{(i)};\quad i = 1, 2;\quad K^{(i)} = \gamma M^{(i)}$$

$$r^{(i)}\ddot{\varphi}^{(i)} = a_\varphi^{(i)} - 2\dot{r}^{(i)}\dot{\varphi}^{(i)} - \psi^{(i)};\quad \varphi^{(i)} = \varphi^{(i)} + \omega t$$

$$R^{(i)} = \frac{K^{(2)}}{\left[r^{(2)}\right]^2}\cos\left[\psi^{(2)} - \psi^{(i)}\right];\quad R^{(i)} = \frac{K^{(i)}}{R_0^2}\cos\psi^{(i)} - \frac{K}{\left[r^{(i)}\right]^2}\cos\left[\psi^{(2)} - \psi^{(i)}\right] \qquad (25)$$

$$\psi^{(i)} = \frac{K^{(2)}}{\left[r^{(2)}\right]^2}\sin\left[\psi^{(2)} - \psi^{(i)}\right];\quad \psi^{(i)} = \frac{K^{(2)}}{R_0^2}\sin\psi^{(i)} - \frac{K^{(1)}}{\left[r^{(i)}\right]^2}\sin\left[\psi^{(2)} - \psi^{(i)}\right]$$

$$\dot{a}_\varphi^{(i)} = \frac{1}{r^{(i)}}\left(a_\varphi^{(i)} v_r^{(i)} - 2a_r^{(i)} v_\varphi^{(i)} + \nu^{(i)}\right);\quad \dot{\nu}^{(i)} = a_\varphi^{(i)}\psi_{\varphi(i)}^{(i)\prime} + a_r^{(i)} R_{\varphi(i)}^{(i)\prime},$$

$$\dot{a}_r^{(i)} = \frac{1}{v_r^{(i)}}\left\{\frac{1}{2}\left[\left(a_r^{(i)}\right)^2 + \left(a_\varphi^{(i)}\right)^2\right] + a_r^{(i)}\left[\frac{\left(v_\varphi^{(i)}\right)^2}{r^{(i)}} - \frac{K^{(i)}}{\left[r^{(i)}\right]^2}\right] - \lambda^{(i)} - \nu^{(i)}\frac{v_\varphi^{(i)}}{r^{(i)}}\right\},$$

$$\dot{\lambda}^{(i)} = -v_r^{(i)}\left(a_\varphi^{(i)}\psi_{r(i)}^{(i)\prime} + a_r^{(i)}\omega R_{r(i)}^{(i)\prime} + \frac{1}{r^{(i)}}a_\varphi^{(i)}\psi^{(i)}\right) - \frac{\nu^{(i)}}{r^{(i)}}\psi^{(i)} - a_r^{(i)} R^{(i)} - \frac{2v_r^{(i)}}{r^{(i)}}\left(\nu^{(i)} - 2a_r^{(i)}\psi^{(i)}\right)$$

$$\left(v_r^{(i)} = \dot{r}^{(i)},\ v_\varphi^{(i)} = r^{(i)}\dot{\varphi}^{(i)}\right)$$

$$J = \int_0^T \left\{\left[\ddot{r}^{(i)} - r^{(i)}\left(\dot{\varphi}^{(i)}\right)^2 + \frac{K^{(i)}}{\left[r^{(i)}\right]^2} + R^{(i)}\right]^2 + \left[r^{(i)}\ddot{\varphi}^{(i)} + 2\dot{r}^{(i)}\dot{\varphi}^{(i)} + \psi^{(i)}\right]^2\right\} dt$$

In a central field (i = 1) the system is simplified [139], [140]:

$$i = 1;\ K^{(2)} = 0;\ r = v_r;\ r\dot{\varphi} = v_\varphi;\ \dot{v}_r = a_r + \frac{v_\varphi^2}{r} - \frac{K}{r^2};\ \dot{v}_\varphi = a_\varphi - \frac{v_r v_\varphi}{r};$$

$$\dot{\lambda} = 0;\ \dot{\nu} = 0$$

$$a_r = \frac{1}{v_r}\left[\frac{1}{2}\left(a_r^2 + a_\varphi^2\right) + a_r\left(\frac{v_\varphi^2}{r} - \frac{K}{r^2}\right) - \lambda - \nu\frac{v_\varphi}{r}\right];$$

$$\dot{a}_\varphi = \frac{1}{r}\left(a_\varphi v_r - 2a_r v_\varphi + \nu\right) \qquad (26)$$

The parameters ν and λ are the integrals of the set; they play the role of the constants of the isoperimetricity with respect to the angular motion (at $\nu = 0$, $\Delta\varphi$ - opt) and the time of travel (at $\lambda = 0$, T - opt). The zero radial velocity point $v_r = 0$ is singular of the node type [12] which is to be accounted for during the computation of the transfer between the circular orbits.

$$v_r = 0:\quad \frac{1}{2}\left(a_r^2 + a_\varphi^2\right) + a_r\left(\frac{v_\varphi^2}{r} - \frac{K}{r^2}\right) - \lambda - \nu\frac{v_\varphi}{r} = 0 \qquad (27)$$

The related set of the equations is invariant to the transformation, connected with the sign change (therefore the foward flight may be replaced by the return flight) and to the transformation, connected with multiplying by ℓ, which permits the variables to be introduced in such a way that the gravitational constant $\bar{K} = 1$:

$$t \to -t;\ \vartheta \to -\vartheta;\ v_r \to -v_r;\ a_\vartheta \to -a_\vartheta;$$
$$r \to \ell_2;\ t \to \ell^{3/2} t;\ v^\circ \to \ell^{-1/2} v^\circ;\ a \to \ell^{-2} a;$$
$$\lambda \to \ell^{-1} \lambda;\ \nu = \ell^{-5/2} \nu;\ \bar{r} = \frac{r}{r_0};\ \bar{t} = \frac{t}{r_0^{3/2}/K^{1/2}};\ \bar{K} = 1 \qquad (28)$$

For the characteristic phases of the space flight two basic problems are formulated: (1) Orbital transfer (with the specified waiting time T_c within the influence sphere of the destination planet 2) escape from the gravitational field (by increasing the energy to zero, $E = 0$) $v_\vartheta(T) = v_{\vartheta_1}$,

$r(0) = r_0;\ \vartheta(0) = 0;\ v_r(0) = v_{r_0};\ v_\vartheta(0) = v_{\vartheta_0} \to r(T) = r_1;\ \vartheta(T) = \vartheta_1;\ v_r(T) = v_{r_1}$;

1. $0 \div 1 \div 0$: $T_c = \dfrac{\vartheta_1 + \vartheta_2 - \omega_1 (T_1 + T_2) + 2n\pi}{\omega_1 - \omega_2}$; $T_\Sigma = T_1 + T_c + T_2$

$\vartheta_1 + \vartheta_2$ и $T_1 + T_2$

2. $r(0) = r_0;\ \vartheta(0) = 0;\ v_r(0) = v_{r_0};\ v_\vartheta(0) = v_{\vartheta_0}$ (29)

$E(T) = \frac{1}{2}\left(v_{r_1}^2 + v_{\vartheta_1}^2\right) - \frac{K}{r_1} = 0$

$\delta J = 0$; 1) $\nu_1 = \nu_2$; $\lambda_1 = \lambda_2$; 2) $\nu = 0$; $\frac{a_{r_1}}{a_{\vartheta_1}} = \frac{v_{r_1}}{v_{\vartheta_1}}$;

$a_{r_1} - 2 a_{\vartheta_1} v_{\vartheta_1} v_{r_1} + \frac{r}{K} + \dot{a}_{r_1} v_{r_1} \frac{r_1^2}{K} = 0$

The optimal conditions for the cited problems are given in the last line of the equations [29]. For the optimal transfer between the circular orbits

$$v_{r_0} = v_{r_1} = 0;\ v_{\vartheta_0} = \sqrt{K/r_0};\ v_{\vartheta_1} = \sqrt{K/r_1};$$
$$\frac{1}{2} a_{a,1}^2 - \lambda - \nu \sqrt{\frac{K}{r_{a,1}^3}} = 0;\quad \text{при } \nu = 0 \to a_0 = a_1 \qquad (30)$$

In a forceless field the set of the equations is integrated in the final form [15], [39], [40], [55], [2]:

$$\text{for } k = 0:\ \ddot{\vec{r}} = \vec{a};\ \dddot{\vec{r}} = 0;\ \vec{a}(t)_{opt} = \vec{b}_0 t + \vec{b}_1; \qquad (31)$$
$$\vec{r}(t) = \frac{1}{6}\vec{b}_0 t^3 + \frac{1}{2}\vec{b}_1 t^2 + \vec{b}_2 t + \vec{b}_3$$

In this case in the simulation problem of the specified velocity increment ΔV for the time T_c max G_n the optimal control corresponds to the law $\vec{a}$ = const:

$$\frac{b_{1x}}{v_{x_0}} = \frac{b_{1y}}{v_{y_0}} = \frac{b_{1z}}{v_{z_0}};\quad J_{min} = \frac{\Delta v^2}{T} \qquad (32)$$

In the problem of the optimal travel between two rest points the linear variation a (t) is optimal:

$$L, T:\quad a(t)_{opt} = (6L/T^2)(1 - 2t/T);\quad J_{min} = 12\frac{L^2}{T^3}$$

(33)

at (a) = const $\mathcal{J} = 16\,\frac{\mathcal{L}^2}{T^3}$

that gives the functional by 1/3 lower than at /a/= const. (fig.21).

The set of the equations is integrated in the final form also in the case of the satellite motion around the sphere surface under the influence of the lateral thrust [71], [86]; when the orbit plane turning is caused by low lateral forces, the optimal control corresponds to sinusoidal variation a (t):

for $r \geq 1$ and $a^* \ll 1$ $\quad \ddot{\theta} + \theta = ae$

which gives the functional by 19 per cent lower than at /a/= const.(fig.22)

Because of the complexity of the basic set of the equations in a number of works the approximate solution to the variational problem is considered. In the problem of the escape from the gravitational field along the gently sloping spiral with low lateral jet acceleration, in which the condition or gravitational and centrifugal acceleration balance is carried out, there shown the optimality of the motion with the constant tangential acceleration /15/, /125/, /121/, /4/: $\frac{v_x^2}{r} \simeq \frac{\kappa}{r^2}$; $\dot{v}_x = a_x$; $\dot{v}_x = a_x$; $\dot{v}_y = a_y - \frac{v_x v_y}{r}$

$$v_x = 2\sqrt{\frac{r_0^3}{\kappa}}\,\frac{a_y}{Q^3};\quad v_y = \sqrt{\frac{\kappa}{r_0}}\,Q;\quad \tau = \frac{t}{r_0};\quad Q = 1 - \frac{\int_0^t a_y dt}{\sqrt{\kappa/r_0}}$$

$$v_{x_0} = 2\sqrt{\frac{r_0^3}{\kappa}}\,a_y;\quad \tau \sim \frac{\sqrt{\kappa/r_0}}{a_y};\quad v_x \sim a_y;\quad a_x \sim a_y^2 \qquad (35)$$

$$\mathcal{J} = \int_0^T a^2 dt = \int_0 a_y^2\,dt \quad (a_x \ll a_y)$$

$$\text{Opt } a_y = \text{const}$$

In fig.23 [140] the examples of such motions are shown, and in fig.24 there shown the required values of the functional for the earth gravitational field.

For the approximate solution of the optimal orbital transfer problem the technique, based on the application of the proposed by Eneev T.M. "transporting" coordinate system , is effective, where as a basic trajectory the Kepler's one is used $x^0(t)$, $y^0(t)$ /3/ /5/ :

$$x^{(0)}(t),\ y^{(0)}(t):\ x = x^{(0)} + \bar{x},\ y = y^{(0)} + \bar{y};\quad z = \bar{z}$$

$$\ddot{x} = a_x;\ \ddot{y} = a_y;\ \ddot{z} = a_z;\ \ddot{a}_x = 0;\ \ddot{a}_y = 0;\ \ddot{a}_z = 0$$

$$\bar{x}(0) = \bar{y}(0) = \bar{z}(0) = \bar{x}(T) = \bar{y}(T) = \bar{z}(T) = 0 \qquad (36)$$

$$\dot{\bar{x}}(0) = \dot{x}_0 - \dot{x}_0^{(0)} \ldots$$

$$\dot{\bar{x}}(T) = \dot{x}_1 - \dot{x}_1^{(0)} \ldots$$

The examples of the application technique of the optimal Earth - Mars transfer are shown in fig.25.

The exact optimal trajectories and the laws of jet acceleration control are found by the numerical integration of the above shown set of the equations. In the problem of escape from the gravitational field at relatively high jet accelerations [20б] the variation of a_y and a_x is of oscillating character (fig.2б). Dimensionless variables permit to obtain the universal relationship between the functional and the time of escaping from the field [34] (fig.27).

Accordingly, the computation of the optimal trajectories and laws a (t) for orbital transfers is made; a number of examples for the optimal

Earth-Mars transfer between the circular co-planar orbits /54/ are given in fig.28. The family of the curves of the optimal values of the functionals J with respect to two parameters: the angular movement and the time of transfer T - permits to determine all the parameters of the optimal transfer (fig.29).

The relationship between the functional, the different dates of the launch and the flight time T is presented in fig.30 with an account for the ellipticity of the co-planar Earth's and Mars' orbits. The interception points of the two families of curves the points of nonunique-ness of the solution to the boundary value problem, in which the transfer time T, the initial and final coordinate values and the velocites and the functional values coincide, - are of interest (see the example in fig.31).

In the problem of the orbital transfer with return and with a specified time T of the flight about the destination planet the optimal conditions for both forward and return flight trajectories are (as it was indicated above) the equalities of the parameters N and along the both trajectories. This condition is satisfied by symmetrical transfers, however, the examples of asymmetrical transfer optimality have been proved (see, for example fig.32 [10б]. The influence of the Z - component of the Earth-Mars -Earth transfer [10б] (fig.33) on the total functional appeared to be gentle, although along the flight stages the functional is essentially different

		Three-dimensional motion - elliptical orbits	Two - dimensional - motion - circular orbits
З - М	T_1 = 184с	J_1 = 6,58	12,91
М - З	T_2 = 512с	J_2 = 24,99	19,02
З - М З	T_3 = 544с , T_6 = 48с	$J_1 = J_2$ = 31,57	31.93

By the certain optimal laws of the jet acceleration a (t) control along the trajectory there can be determined also the remaining parameters: the thrust control $P(t)$, consumption control $q(t)$ exhaust velocity control $V(t)$, the vehicle weight variation $G(t)$ along the trajectory:

$$\text{opt } a(t) \rightsquigarrow N, P, q \text{ и } V: \quad N = \frac{G_н}{\alpha}; \quad \frac{G_н}{G_0} = \sqrt{\Phi_1} - \Phi_1$$
$$\Phi_1 = \Phi(T), \quad \Phi(t) = \frac{\alpha}{2g}\int_0^t a^2 dt$$
$$P(t) = G(t)\frac{a(t)}{g}; \quad q(t) = G^2(t)a^2(t)/2g^2N \qquad (38)$$
$$V(t) = 2Ng/G(t)a(t); \quad G(t) = G_0/\left(1 + \frac{G_0\Phi(t)}{G_н}\right)$$

The example shown in fig.34 for the Earth - Mars - Earth mission [16б] suggests that considerable exhaust velocity control is necessary for the optimal program.

As a whole the consideration of the ideally - controlled propulsion system permited to determine the upper boundary of the pertinent class propulsion system possibilities. To evaluate the maximum deterioration of

the characteristics, caused by the restrictions on the regulation, another limit case is considered - the power limited unregulated propulsion system: the system with constant thrust, consumption, exhaust velocity, but with the possibility of thrust out-off [30] and others.

$P = const$, $q = q_0$; $V = V_0$ $(N = const)$; $G_\Sigma = G_M + G_N$

$\dot{G}_\Sigma = -q_0\delta$; $\vec{\tau} = \vec{v}$; $\dot{\vec{v}} = \sqrt{\frac{2P}{G_N q_0 \delta}}\,\vec{i} + \vec{R}$;

$G_\Sigma + G_N$

$\dot{G}_N = 0$; $\dot{q}_0 = 0$ $\quad \left|\begin{matrix}\delta = 1 \\ \delta = 0\end{matrix}\right.$ (39)

$\delta(t)$, $\vec{i}(t) \rightsquigarrow G_\Sigma(0) + G_N = 1$; $\vec{r}(0) = \vec{r}_0$; $\vec{v}(0) = \vec{v}_0$;

$\vec{r}(T) = \vec{r}_1$, $\vec{v}(T) = \vec{v}_1$

for max $G_\Sigma(T)$

For the particular variational problem solution it is necessary to formulate such controls $\delta(t)$ and $\vec{i}(t)$ and choose such values of the constant parameters $\bar{G}_N$ and q, which provide $\bar{G}_{\Sigma\,max}$. The weight and trajectory ingradients of the problem are not separated.

The problem is essentially simplified for the homogenous gravitational field, for which the solution for the optimal thrust direction is following [30], [38], at $\vec{R}(\vec{r}, t) = const$, opt $\vec{i}(t) = \frac{\vec{b}_2 - \vec{b}_1 t}{\sqrt{b_2^2 - 2\vec{b}_2\vec{b}_1 t + b_1^2 t^2}}$

$$\Phi_L = \frac{\alpha}{2g}\,\frac{12L^2}{T^3} \qquad (40)$$

For the motion between the two rest points the optimal solution depends on the functional $\Phi_L = \frac{\alpha}{2g}\frac{12L^2}{T^3}$ (see fig.35). The optimality of one coasting are at the trajectory center of the duration 1/3 T is shown.

The results of the numerical solution to the problem of the optimal orbital transfer Earth-Mars with P = const. [165] are presented in fig.36. The comparison of the functional value for the regimes P_{opt} = var, P = const with power on and off [168] is illustrated in fig.37.

The introduction of the direct restrictions on the propulsion system parameters (the particular case is P = const.) makes the solution to the optimization problem much more complex, since in this case the trajectory and weight ingradients of the problem are not separated. The mentioned difficulties do not exist, if the restrictions are imposed on the acceleration a due to thrust; however these conditions account for the restrictions, imposed on the propulsion system parameters (consumption and jet velocity), only in indirect manner. The solution of the weight ingredient of the problem /19/ (see above) entirely extended to the case of the limited acceleration; the trajectory optimization problem consists as previously of providing the minimum of the functional $J = \int_0^T a^2 dt$.

In the case of constant $a(t) = Q_0$ the problem of minimum is equivalent to that of minimum T at specified value of , the latter for-

mulation is used for the problem solution in a central field /51/, /58/, /126/ (fig.38).

$$a(t)=a_0=const;\quad \mathcal{J}=a_0^2 T;\quad \dot{z}=v\sin\psi;\quad \dot{\varphi}=-\frac{v}{z}\cos\psi;$$

$$\dot{v}=a_0\sin(\psi-\varphi)-\frac{1}{z^2}\sin\psi \qquad (41)$$

$$\dot{\psi}=\frac{a}{v}\cos(\psi-\varphi)+(1-zv^2)\cos\psi/z^2v$$

For near-orbits transfer /126/: at $z_1\approx 1$; $a_0 \ll 1$.

$$Opt\ Ctg\,\psi=\frac{\sin(t+t_0)}{2\cos(t+t_0)+C_0}. \qquad (42)$$

The comparison between the approximate (dotted line) and exact (continuous line) programms (t) for the example of Earth-Mars mission at $a_0 = 0{,}1$ cm/sec^2 (T = 186 . 24 hours) is given in fig.39. Here the value $\mathcal{J} = 16{,}1$ m^2/sec^3, instead of the value 6,6 m^2/sec^3 is representative according to the optimal solution.

The possibility of the power off:

$$a(t)=a_0 \;\Big|\; \dot{\mathcal{J}}=a_0^2\delta;\ \vec{z}=\vec{r};\ \vec{r}=a_0\vec{i}\,\delta+\vec{R};\ at\ min\ \mathcal{J}(T):$$

$$a(t)=0 \;\Big|\; \vec{i}(t)\,u\,\delta(t);\ \mathcal{J}(0)=0,\ \vec{z}(0)=\vec{z}_0,\ \vec{v}(0)=\vec{v}_0;\ \vec{z}(T)=\vec{z}_1;\ \vec{v}(T)=\vec{v}_1$$

$$at\ \vec{R}=0;\ L,T:\quad \mathcal{J}_{min}=13{,}5\frac{L^2}{T^3} \qquad (43)$$

$$\left(\mathcal{J}_{opt}=12\frac{L^2}{T^3};\ \mathcal{J}_{a=const}=16\frac{L^2}{T^3}\right)$$

improves considerably the functional of the problem in this case. For the motion between the rest points $\mathcal{J} = 13{,}5$ /3/ (see the dotted line in fig.21 and compare with (33)) and a coasting arc takes 1/3 of the total time; for the problem of the orbit plane turning (see the dotted line in fig.22) the departure from the optimal functional $\simeq 8$ per cent and the coasting are cover the arc $\pm 23{,}2^\circ$ /23/. The comparison for the example of the Earth-Mars mission is given in fig.40 /168/. The problem of the approximation of the complex optimal law of limited-power propulsion system control by a simple stepwise law is investigated in detail in /33/.

The ideally regulated and unregulated propulsion systems, considered above, represent the upper and lower criteria of the power - limited regulated propulsion system capabilities. As has been noted above, in a real accelerator, when the jet velocity being regulated in a certain range, the efficiency depends on ; at α = const. (see ref.39 and 168):

$$0\le N(t)\le N_{max}(V)\le N_0;\quad 0\le V_{min}\le V(t)\le V_{max}<\infty$$

$$G_N=\alpha N_0\ (\alpha=const);\quad N_{max}(V)=N_0\,\eta(V);\quad \bar{N}=\frac{N}{N_{max}}: \qquad (44)$$

16 $\dot{G}_\Sigma = -\frac{2g}{\alpha} G_N \varphi(V) \frac{\bar{N}}{V^2}$; $\dot{\vec{r}} = \vec{v}$; $\dot{\vec{v}} = \frac{\frac{2g}{\alpha} G_N}{G_\Sigma + G_N} \varphi(V) \frac{\bar{N}}{V} \vec{i} + \vec{R}$; $\dot{G}_N = 0$

$$G_\Sigma = G_n + G_N|V(t), \ \vec{i}(t) \text{ и } \bar{N}(t) \rightsquigarrow \max G_\Sigma(T) \qquad (44)$$

$$G_\Sigma(0) + G_N(0) = 1, \ \vec{r}(0) = \vec{r}_0, \ \vec{v}(0) = \vec{v}_0; \ \vec{r}(T) = \vec{r}_1, \ \vec{v}(T) = \vec{v}_1$$

For the written set of the equations the optimal controls $V(t)$, $\vec{i}(t)$, $\bar{N}(t)$ and opt $\bar{G}_N$, providing $\bar{G}_{n\,max}$, are determined. As an example, the following simplest regulating characteristic $\varphi(V)$ is considered:

$$\left.\begin{matrix}\bar{N}=1 \\ \bar{N}=0\end{matrix}\right| \varphi(V) = \begin{cases} 1 \text{ at } V_{min} \le V \le V_{max} \\ 0 \text{ at } V \le V_{min}\,; V \ge V_{max} \end{cases}$$

which in limit cases transforms into ideally regulated and unregulated systems (fig.41). The presence of the arcs with $V_{opt}(t)$, $V = V_{max}$ and $\bar{N} = 0$ is seen in (fig.42). In fig.43 indicated the programm of optimal restricted-exhaust velocity control for the example of the Earth-Mars mission. Another kind of restrictions, that is imposed on the time of the propulsion system operation, is considered in [30].

With long time of the propulsion systems operation the problem of reliability becomes of especial importance. The relation of the reliability problem to the choice of the optimal trajectory and weight parameters has been investigated in detail in [17], and accordingly, the account for the trajectory correction in the problems of the weight parameters optimization for the power-limited propulsion systems has been considered in [18].

In conclusion it may be noted, that the propulsion system contrasting, contained in a number of works, is not always scientifically justified. It is more correct to investigate [12], [32] the optimal combination and the domains of the reasonable application of the various propulsion systems.

As an example in fig.44 [12] for the two simulation problems (velocity increment flight along the distance L) there presented data of power-limited propulsion systems combination with limited jet velocity. The shaded domain, where this combination is reasonable, corresponds to the payload proximity for each of the propulsion systems.

The indicated data deal briefly only with the formulating and resulting aspects of the basic carried out works; the statement of the essential investigations by the methods of the variational problem solution see in Part IV of our survey [24].

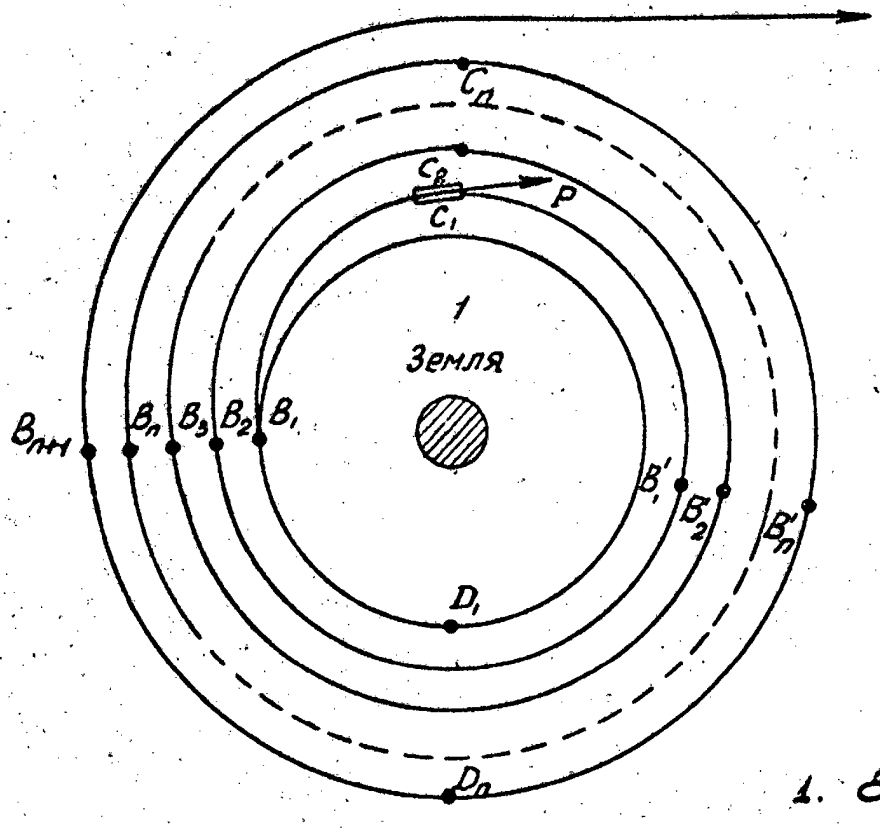
Table 1

Propulsion system type	Principal scheme	Jet composition	Exhaust velocity $V\,[m/sec]$	Specific weight of the propulsion system (Propulsion system weight devieded by maximum thrust) $\gamma = G_{g.y}/P_{max}$	Acceleration due to thrust (Thrust devided by the vehicle mass) $a\,[m/sec^2]$
1	2	3	4	5	6
Thermal jet engines	Chemical combustion chamber	Gaseous combustion products	up to $4.5\ 10^3$	10^{-2}	10 500
	Solid nuclear propellant reactor	Dissociated gas	$8\ 10^3 - 1.2\ 10^4$	10^{-1}	1 100
	Liquid nuclear propellant reactor	Dissociated gas	$1.2\ 10^4 - 2\ 10^4$	10^{-1}	1 100
	Gaseous nuclear propellant reactor	Plasma	$2\ 10^4 - 7\ 10^4$	10^{-1}	1 100
Electrical jet engines	Arc heater	Plasma	up to $2.5\ 10^4$	10^3	10^{-3} 10^{-2}
Power limited engines	Electrostatic accellerator	Plasma	$5.10^4 - 10^6$	10^3	10^{-5} 10^{-2}
	Electrodynamic accelerator	Plasma	$5.10^4 - 10^6$	10^3	10^{-5} 10^{-2}
Solar sail	Mirror surface reflecting solar rays	Photons of the solar radiation	3.10^{-8}	10^4	10^{-5} 10^{-3}

Table II

Planet	Mercury	Venus	Jupiter	Saturn	Uranus	Neptun	Pluto	
T, years	0.53	0.45	6.6	17	49	96	145	

Payload, G_n, m	Mass of the unit sail area, $\rho\delta$, g/cm^2	Sail demensions		Total vehicle weight, G_0, m	Initial acceleration due to solar pressure, a_0 cm/sec^2	Duration of the maneuver T, 24hours	Radius of the initial circular orbit, z_0, Km.	Maximal departure of the trajectory till the take off instant, Km
		Area, S_0, m^2	Equivalent circle diameter, m					
0.1	10^{-4}	10^6	$1.12\ 10^3$	1.1	0.85	54	$7.25\ 10^3$	$4.2 \cdot 10^5$
0.1	2.10^{-4}	10^6	$1.12\ 10^3$	2.1	0.44	112	$7.25\ 10^3$	$6.1\ 10^5$
0.1	3.10^{-4}	10^6	$1.12\ 10^3$	3.1	0.3	119	$7.25\ 10^3$	$7.62\ 10^5$
0.1	10^{-4}	$2.46\ 10^5$	$0.56\ 10^3$	0.546	0.67	73	$7.25\ 10^3$	$4.78\ 10^5$
0.1	10^{-4}	10^5	$0.356\ 10^3$	0.2	0.46	105	$7.25\ 10^3$	$5.87\ 10^5$



1. Earth

Fig.1

1. Sail 2. Solar rays

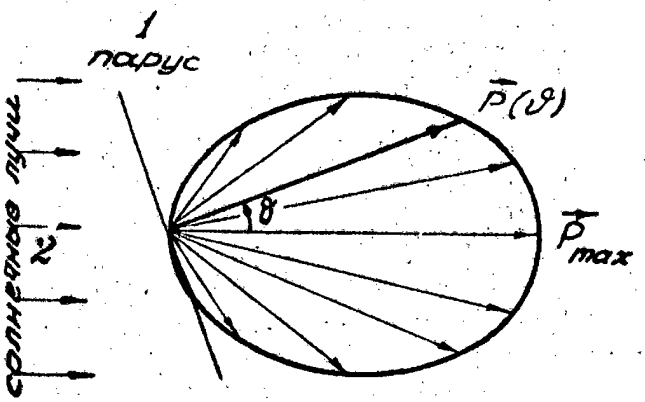


Fig. 2

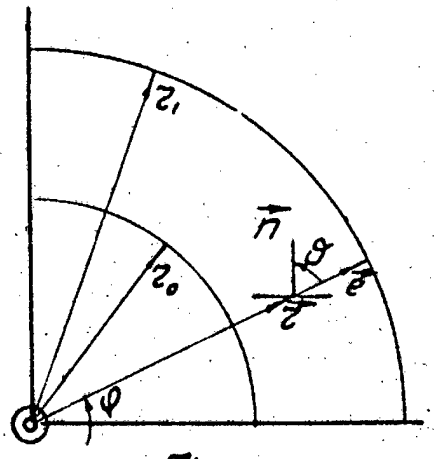


Fig. 3

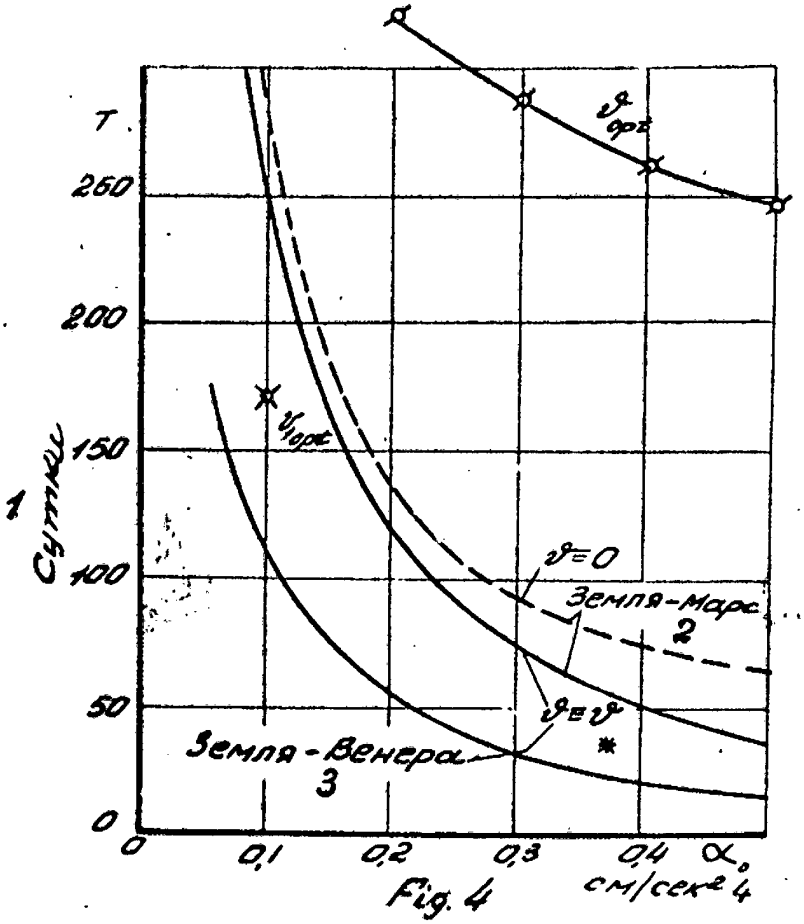


Fig. 4

1. 24 hours
2. Earth - Mars
3. Earth - Venus.
4. sm/sec^2

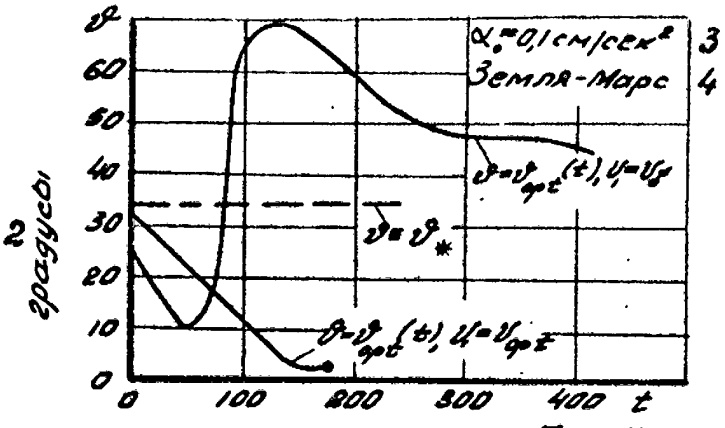


Fig. 5

1. 24 hours
2. Degrees
3. $\alpha_0=0,1\ sm/sec^2$
4. Earth - Mars.

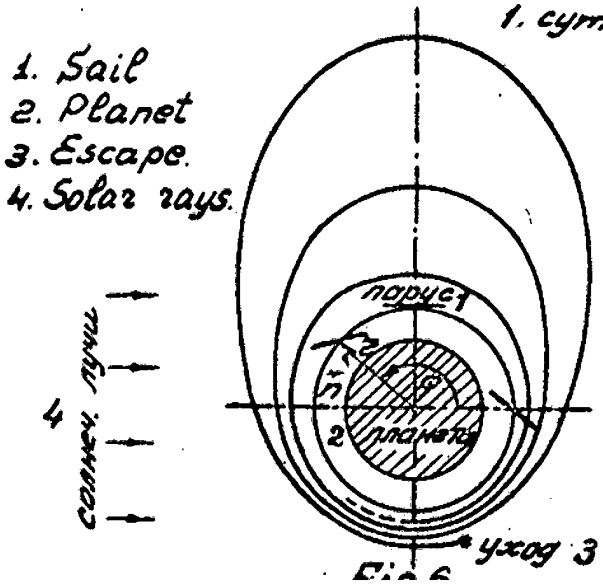


Fig. 6

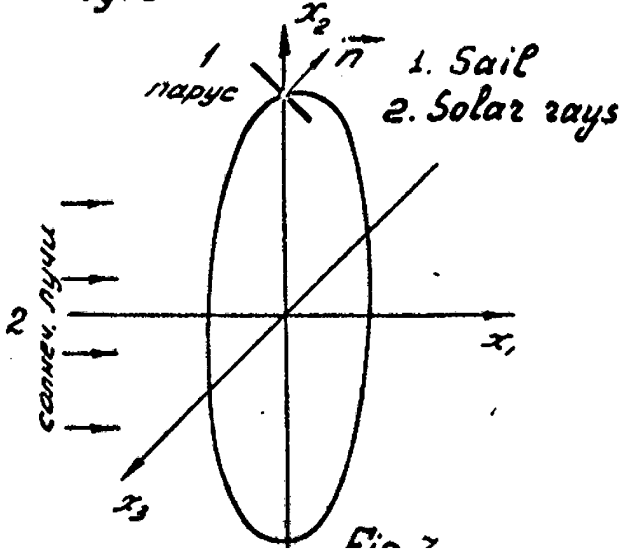
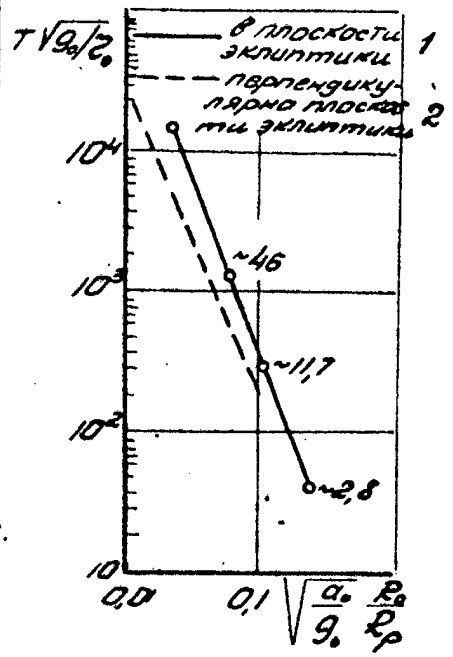


Fig. 7

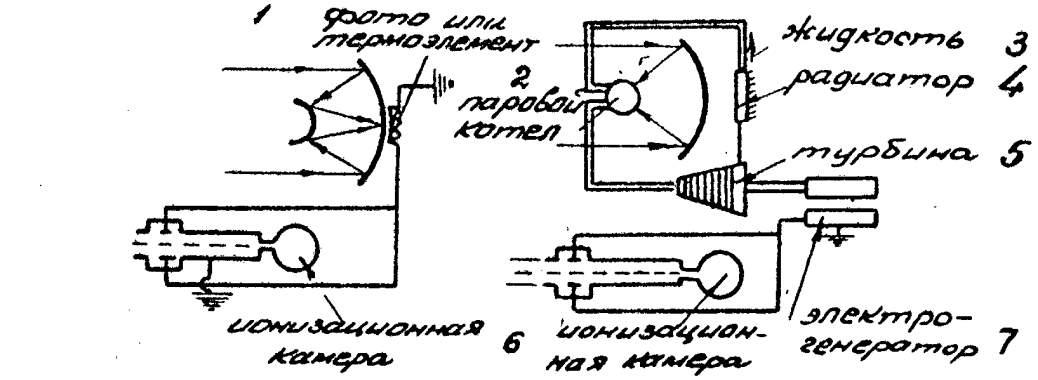
1. Photo or thermo-element.
2. Vapour vessel.
3. Liquid.
4. Radiator
5. Turbine
6. Jonization chamber.
7. Electric generator.
8. Nuclear reactor.
9. High pressure vapour.
10. Turbine
11. Liquid.
12. Radiator.
13. Low pressure vapour
14. Electric generator.



1. In ecliptic plane.
2. Perpendicular to the ecliptic plane.

Fig. 8

15. β - rays



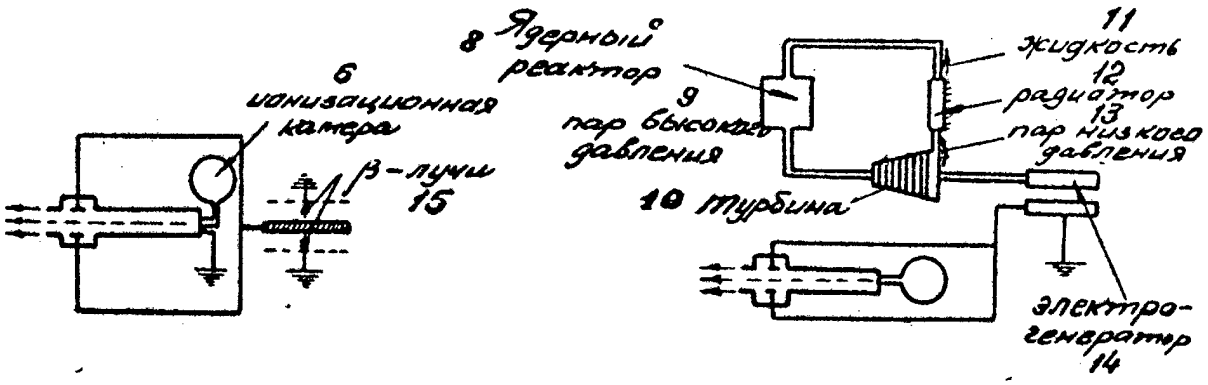


Fig. 9

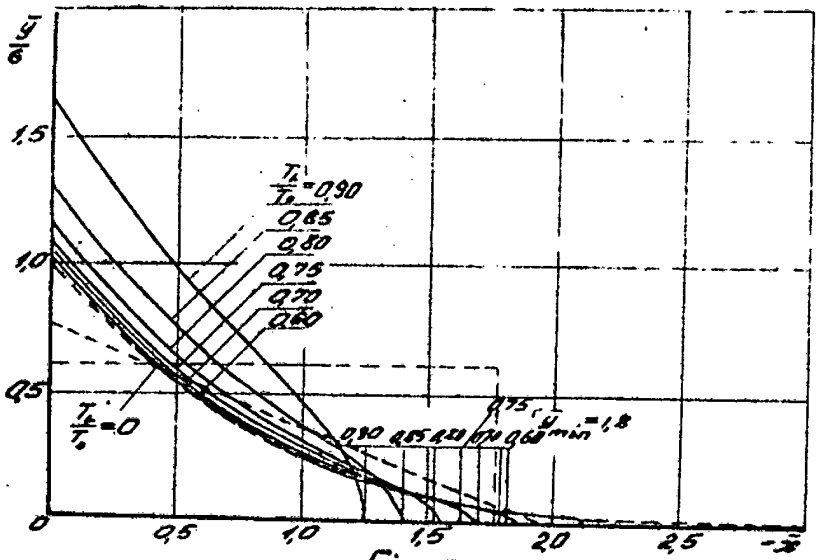


Fig.10

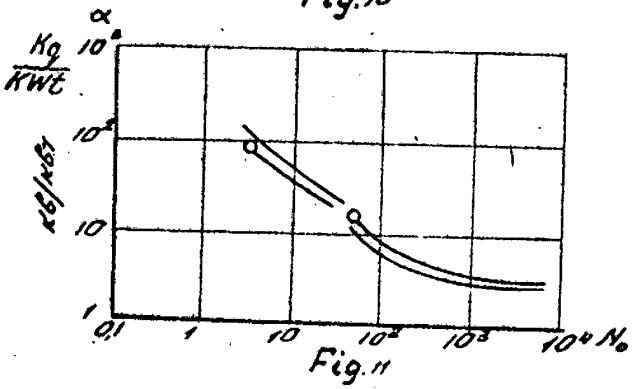


Fig.11

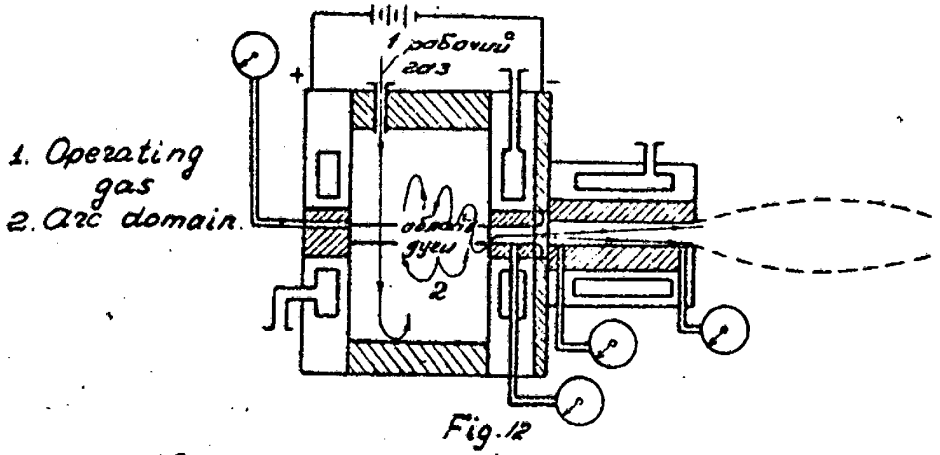
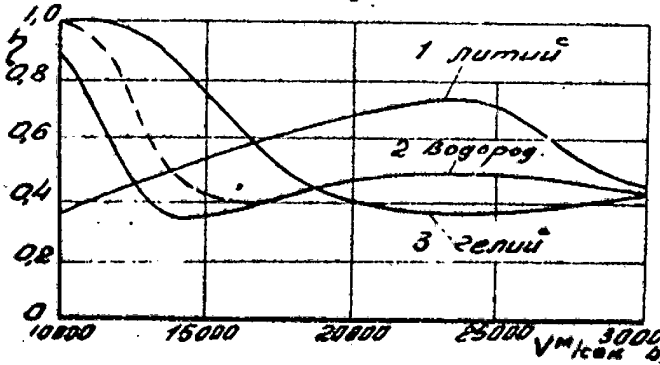
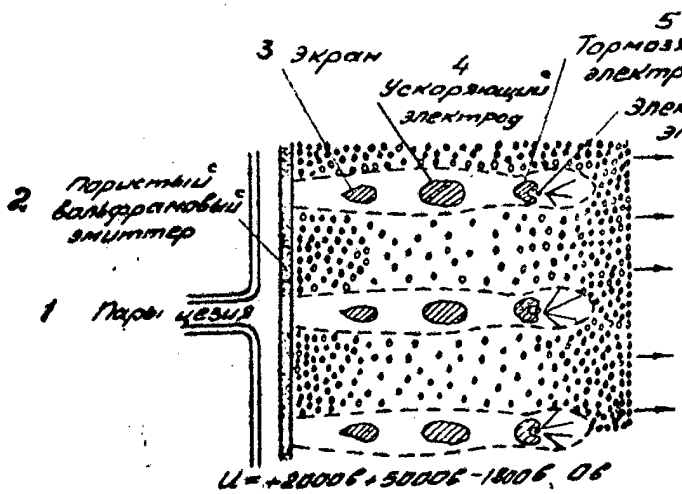


Fig.12



1. Lithium
2. Hydrogen
3 Helium
4. m/sec.

Fig.13



1. Cesium vapour
2. Porous tungsten emitter
3. Screen
4. Accelerating electrode
5. Decelerating electrode
6. Electronic emitter.

Fig. 14

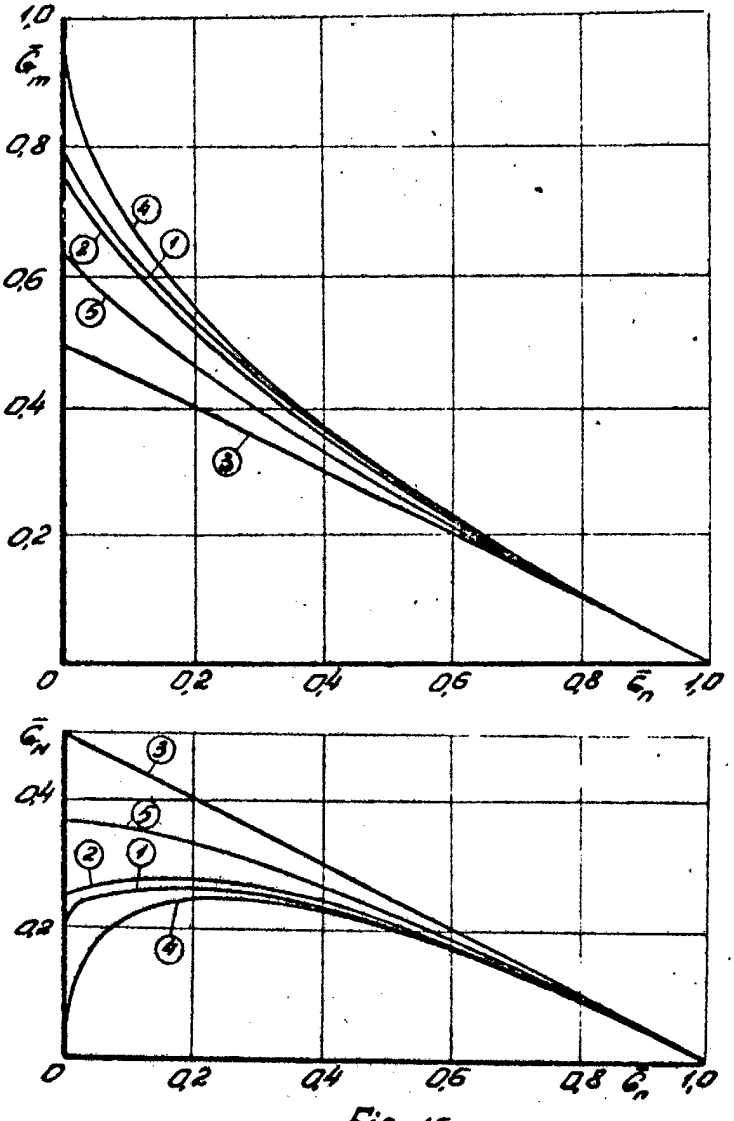


Fig 15

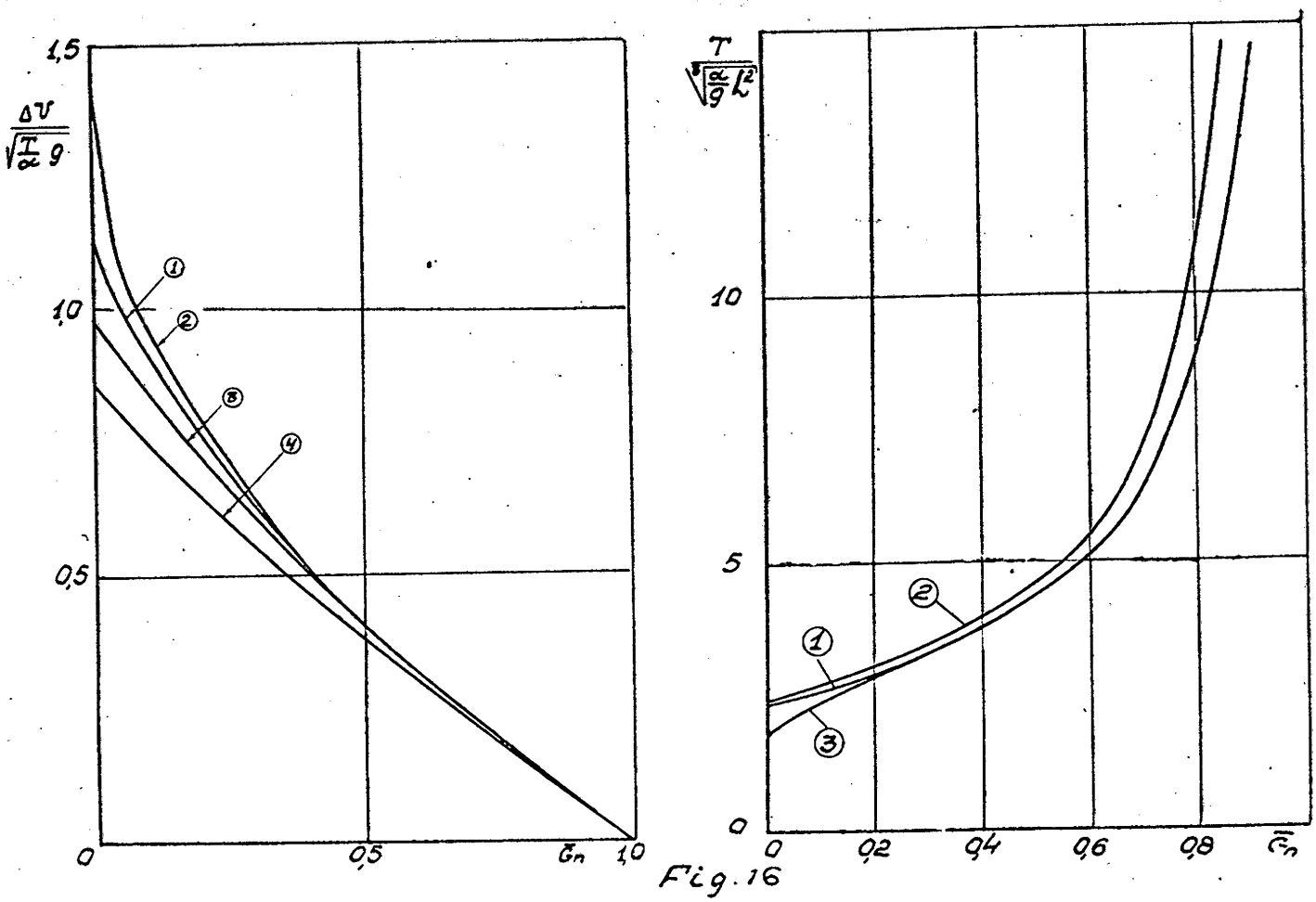


Fig. 16

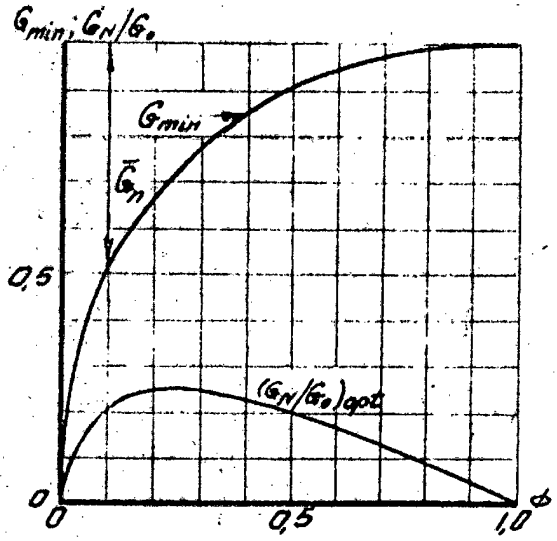


Fig. 17

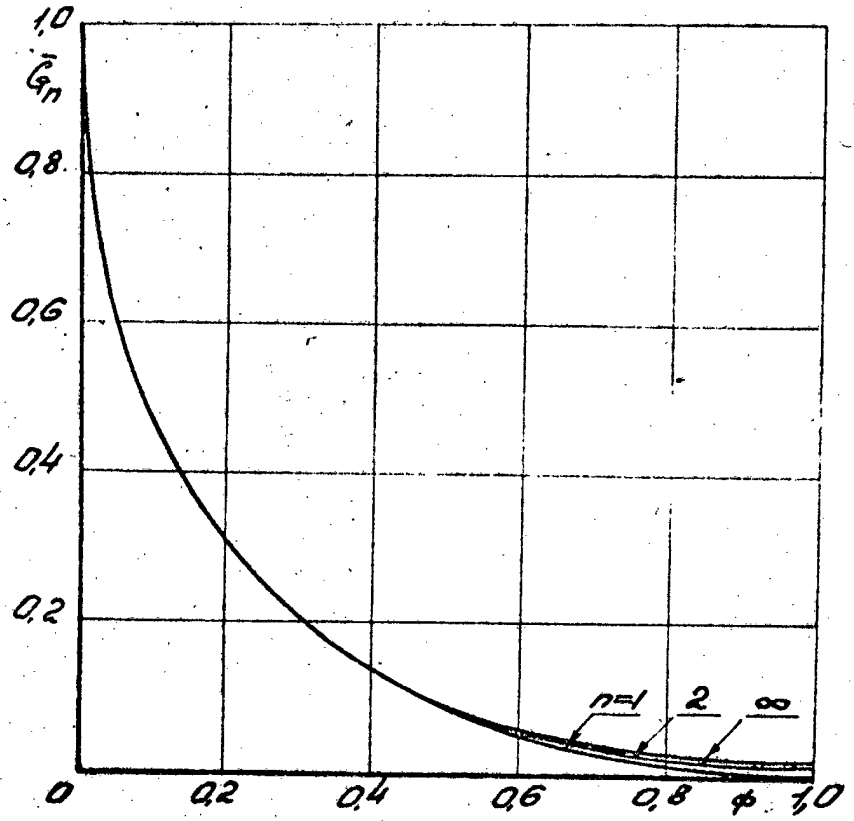


Fig. 18

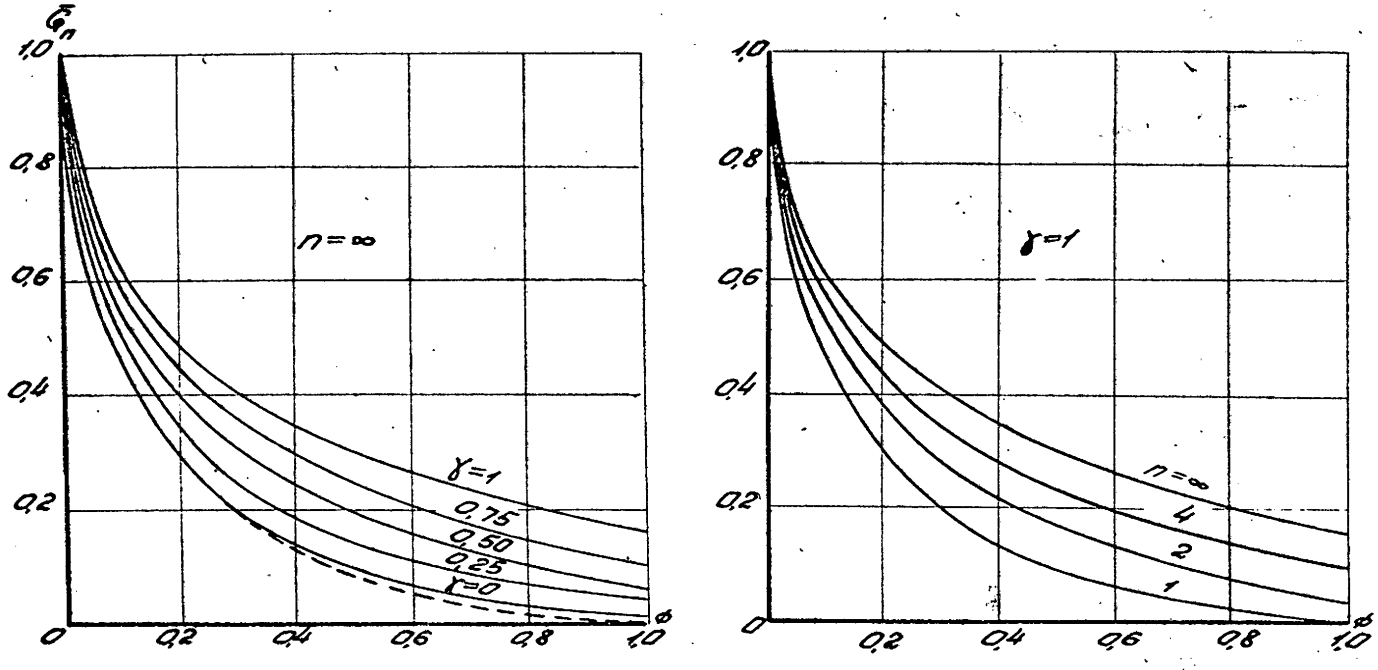


Fig. 19

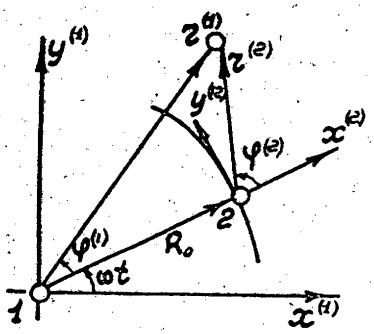


Fig.20

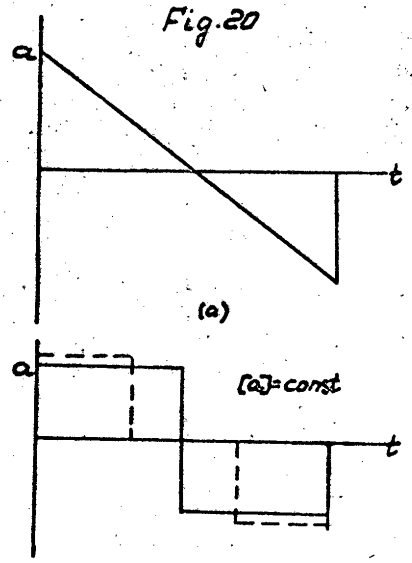


Fig.21

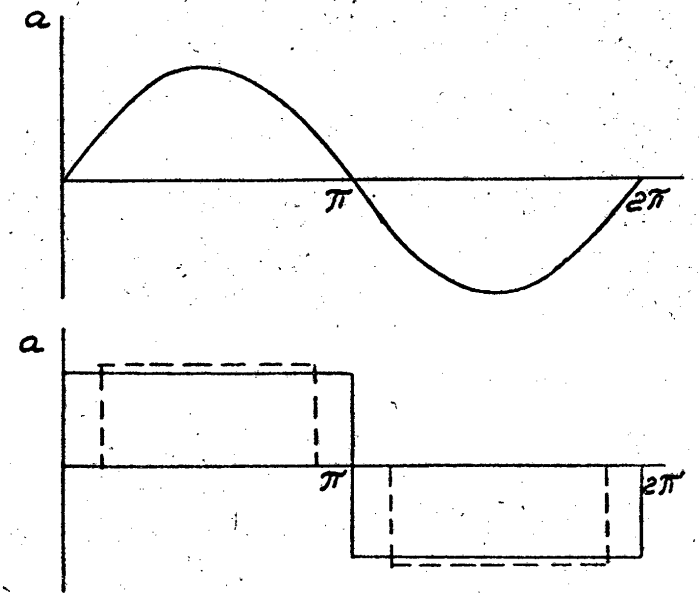
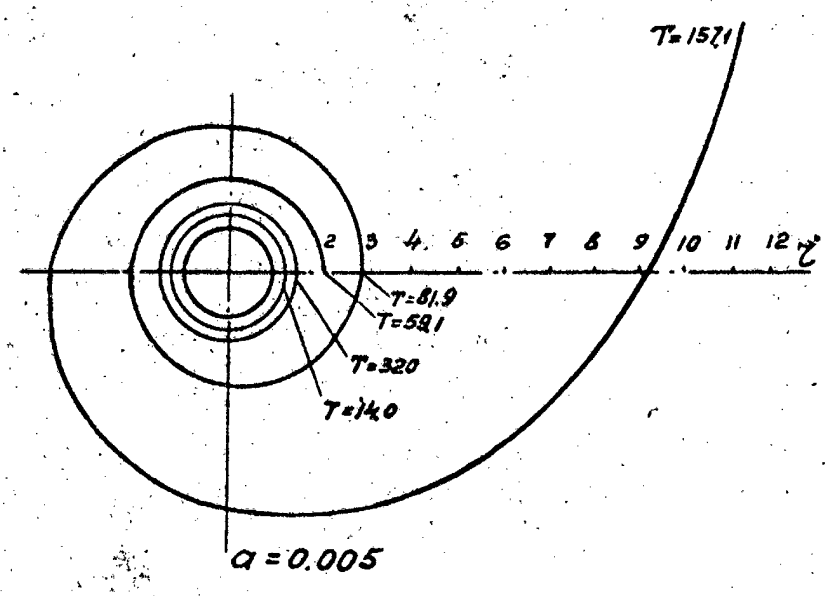
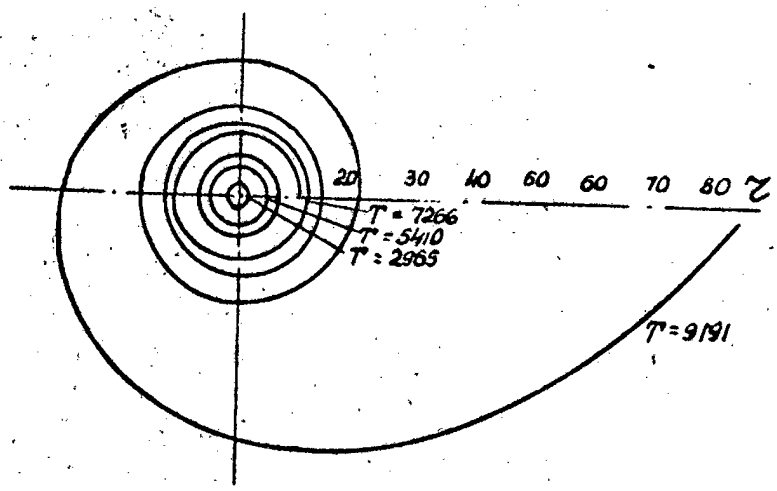


Fig.22

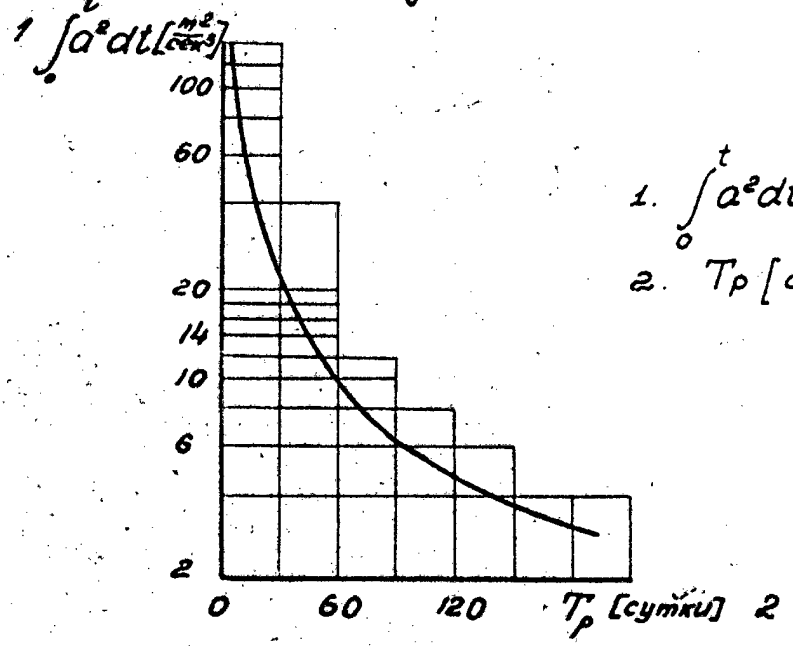


a = 0.005



a = 0.0001

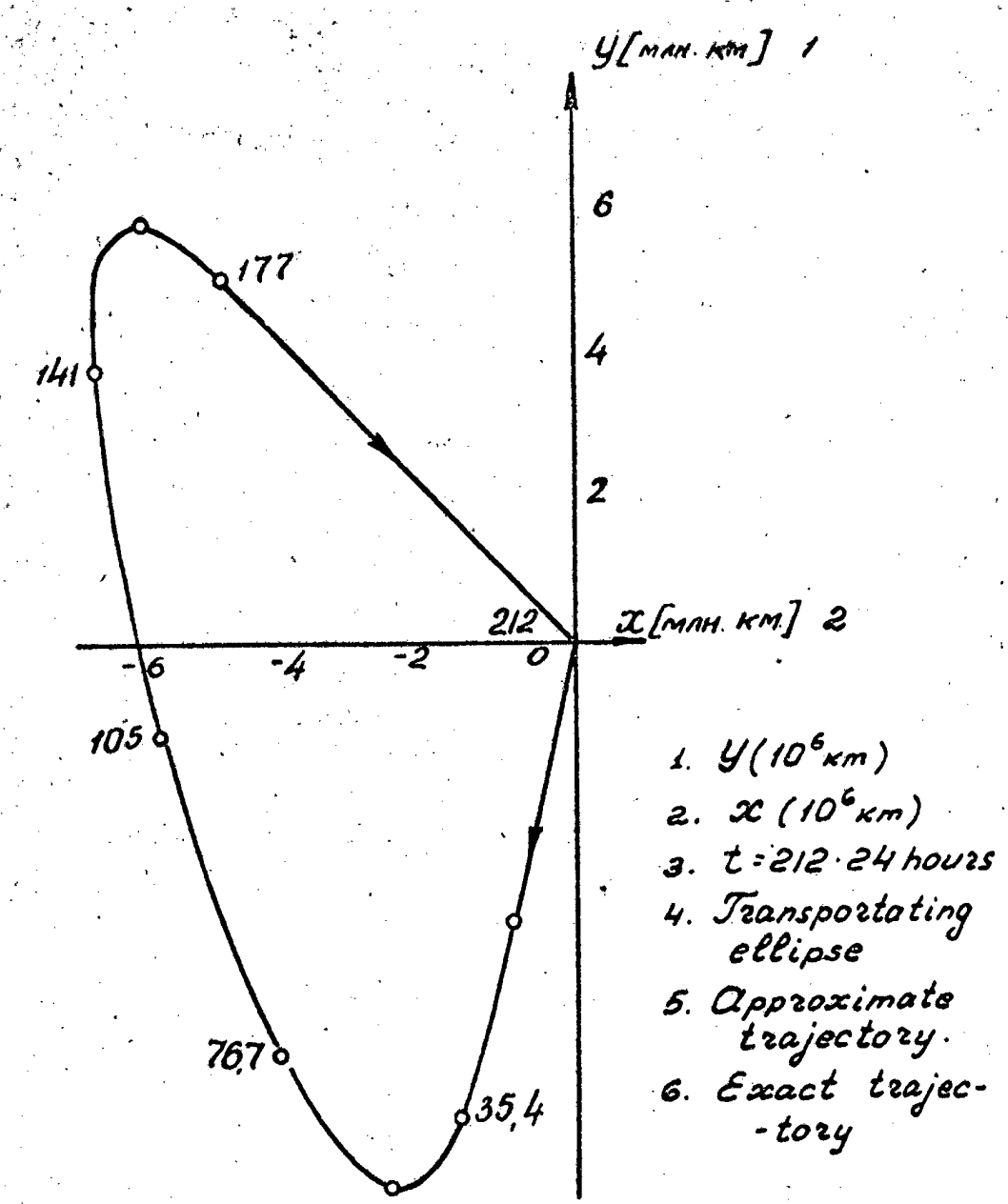
Fig. 23



1. $\int_0^t a^2 dt$ m^2/sec^3

2. T_p [24 hours]

Fig. 24



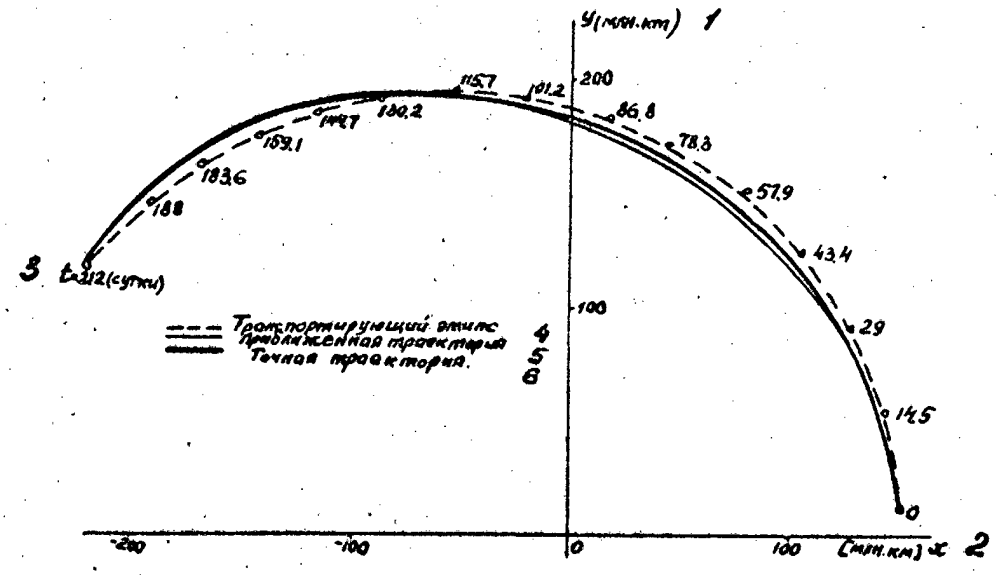


Fig. 25

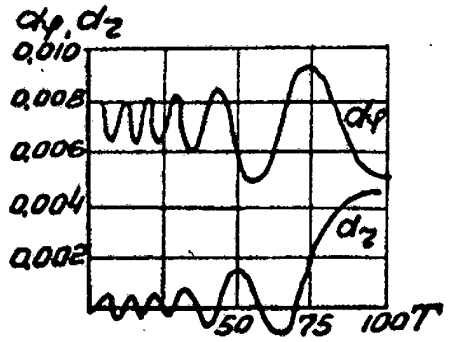


Fig. 26

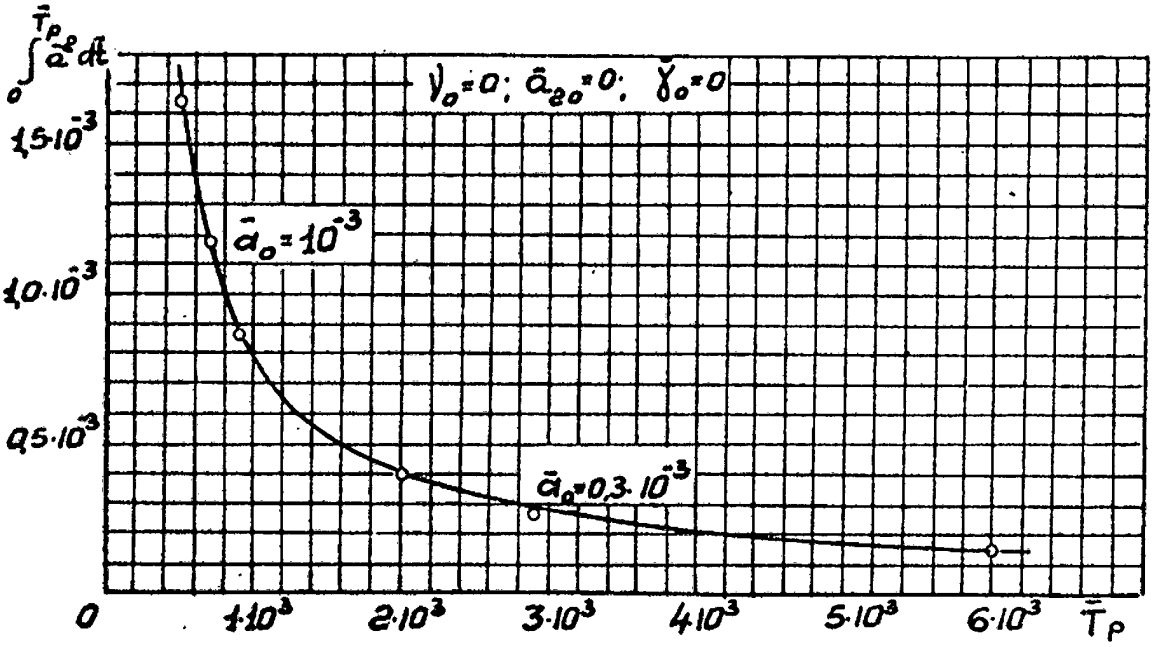
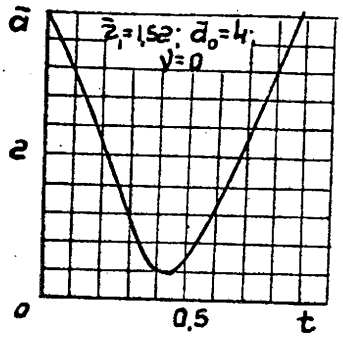
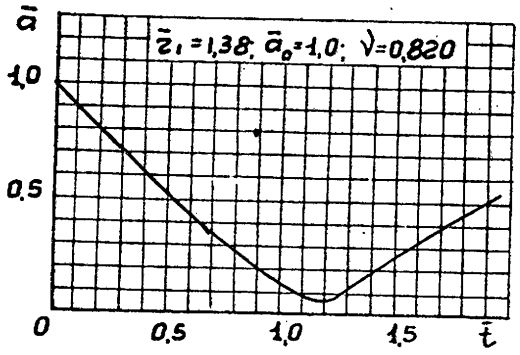
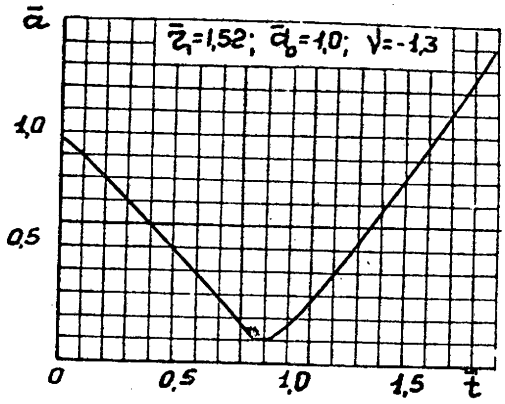
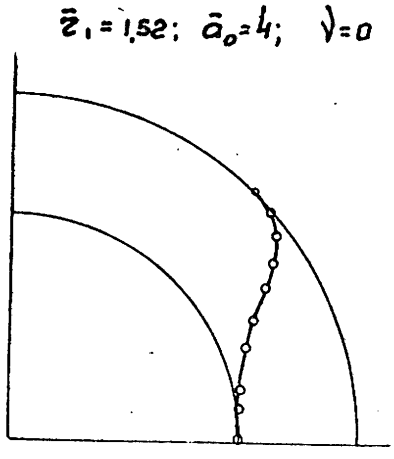


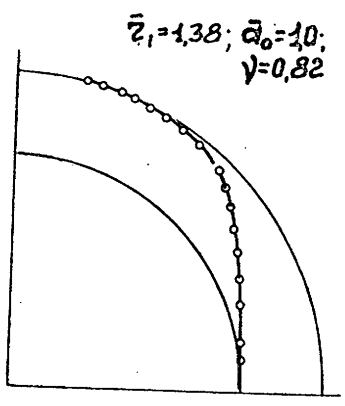
Fig. 27











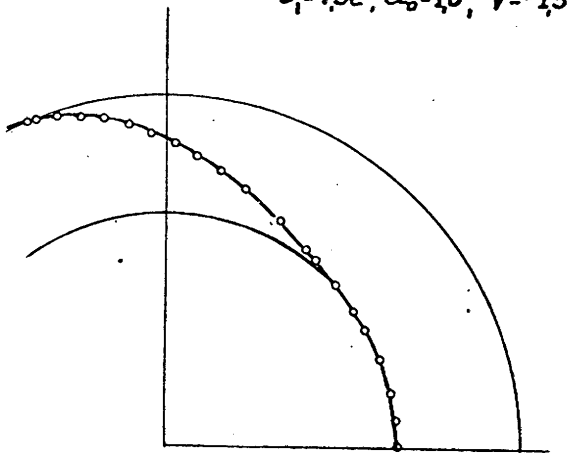


Fig. 28

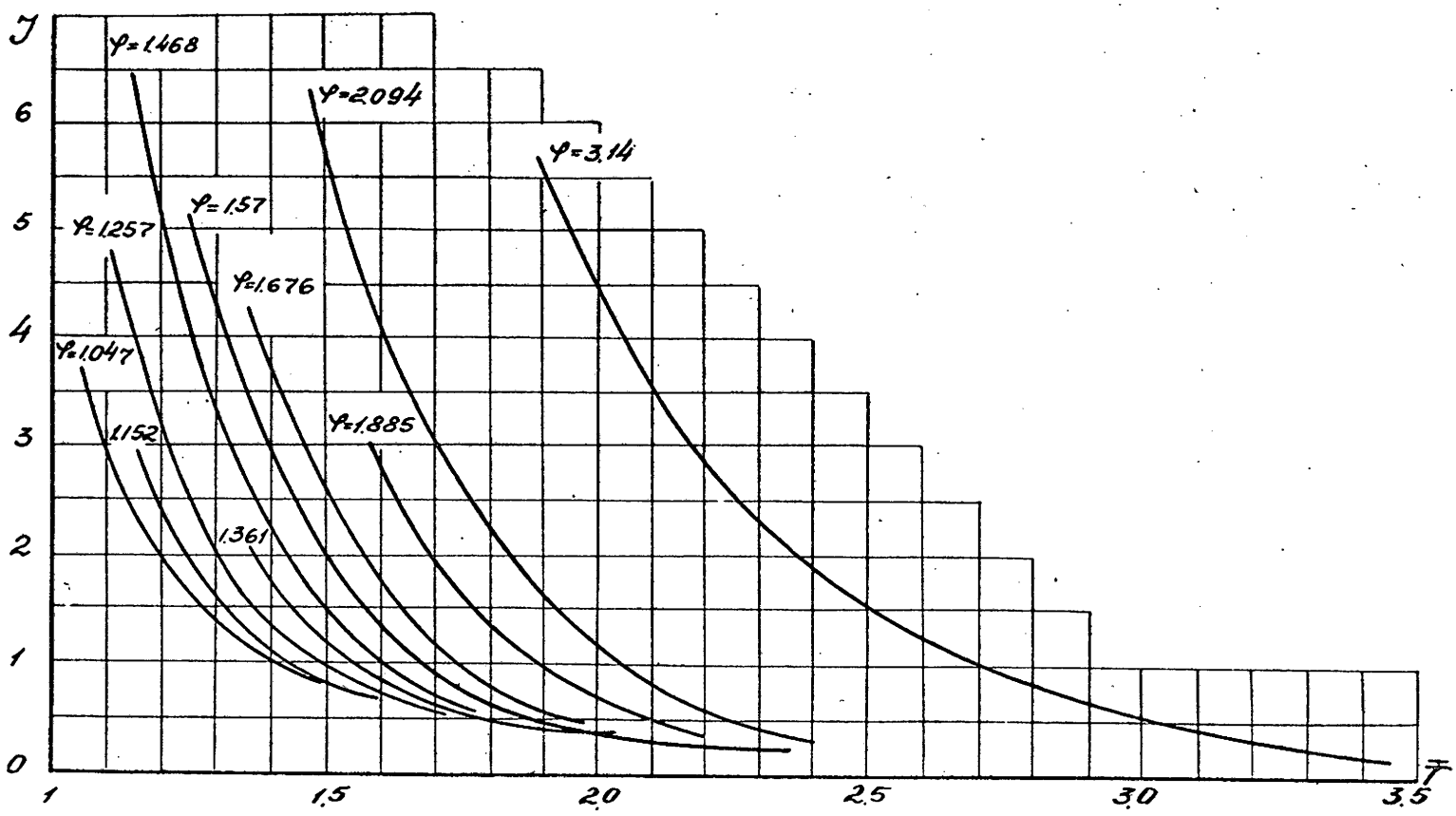
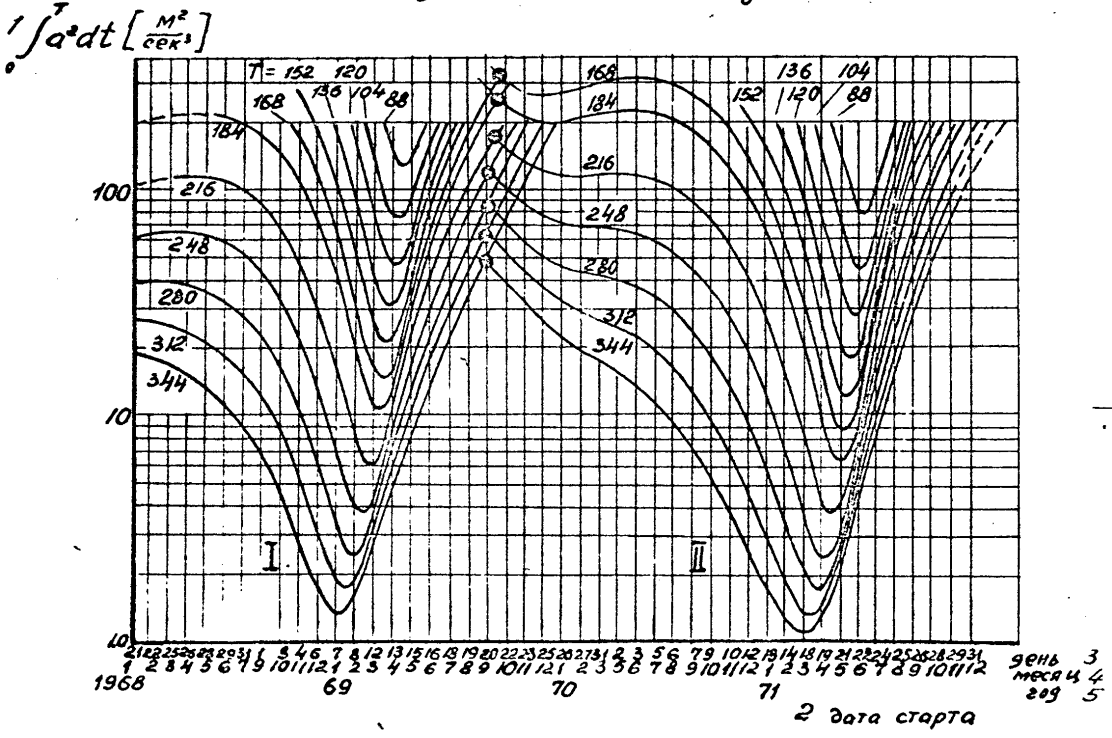


Fig 29

1. $\int_0^T a^2 dt \left[\frac{м^2}{сек^3}\right]$ 2. Launch date.

3. Day. 4. Month 5. Year.



1. Apr. 8.5 1970

2. Oct. 6.5 1969

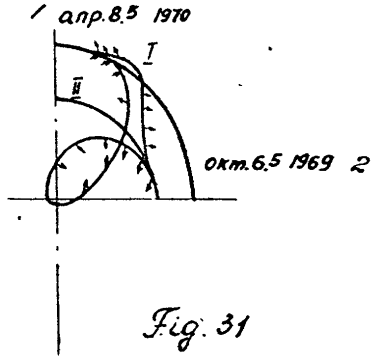


Fig. 31

Fig. 30

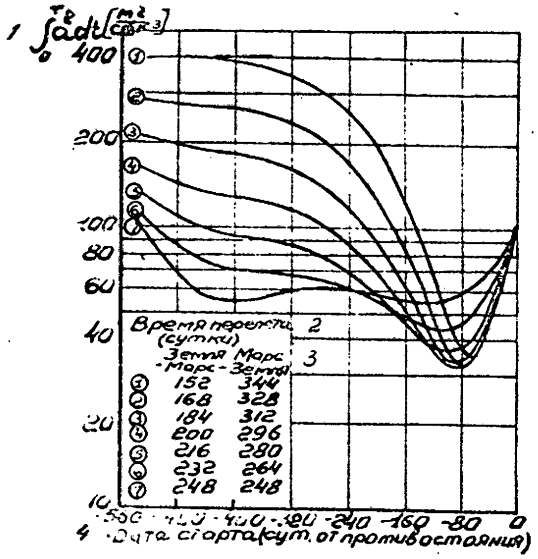
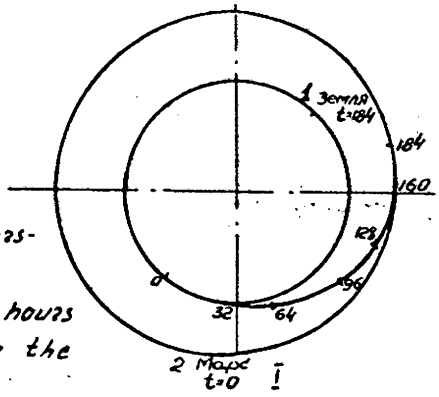


Fig. 32

1. $\int_0^T a^2 dt$ m^2/sec^3
2. Transfer time (24 hours)
3. Earth - Mars - Mars - Earth
4. Launch date (24 hours with reference to the opposition.)



1. Earth.
2. Mars.
3. $T_{opp} = 48 \cdot 24$ hours
4. Earth
5. Mars.

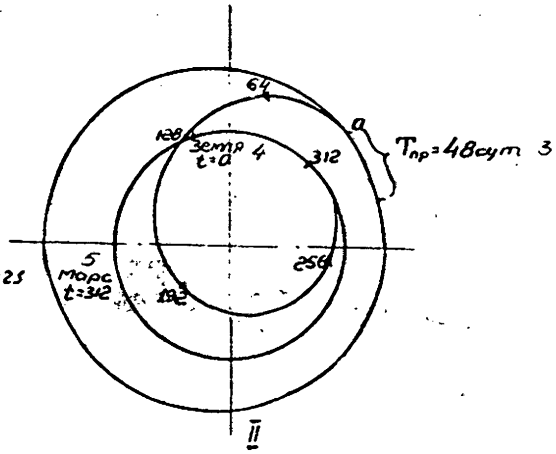


Fig. 33

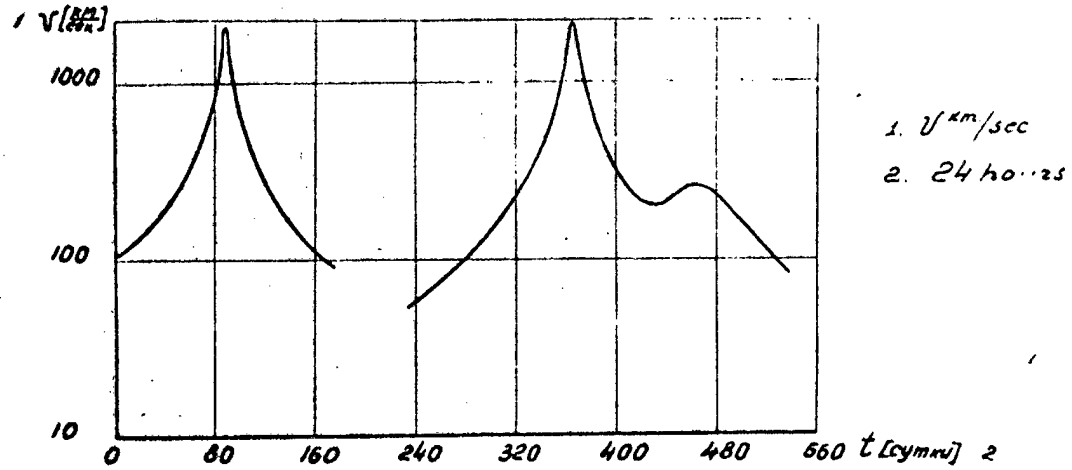


Fig. 34

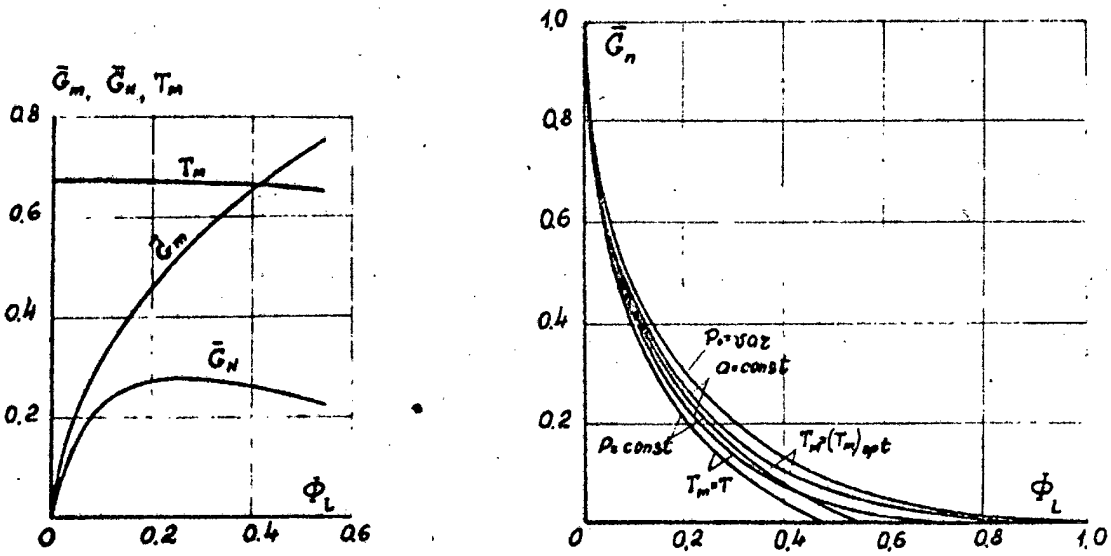
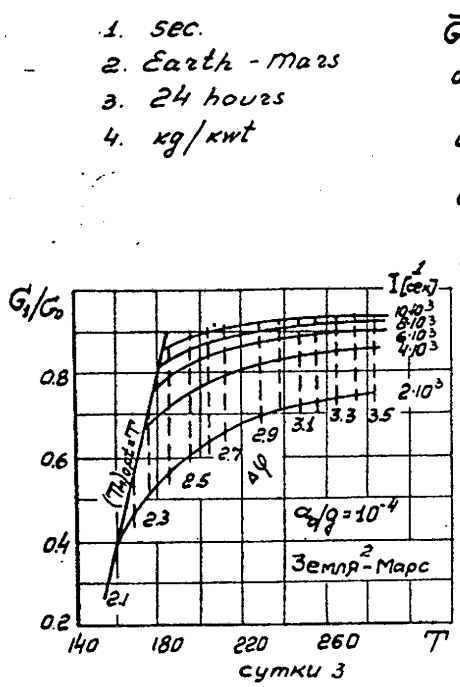


Fig. 35



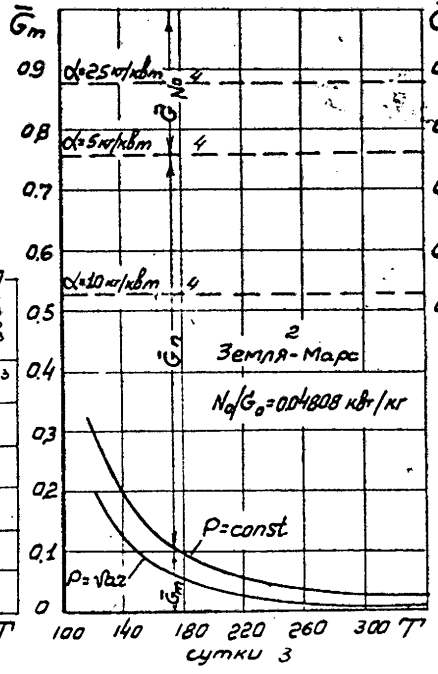


Fig. 36

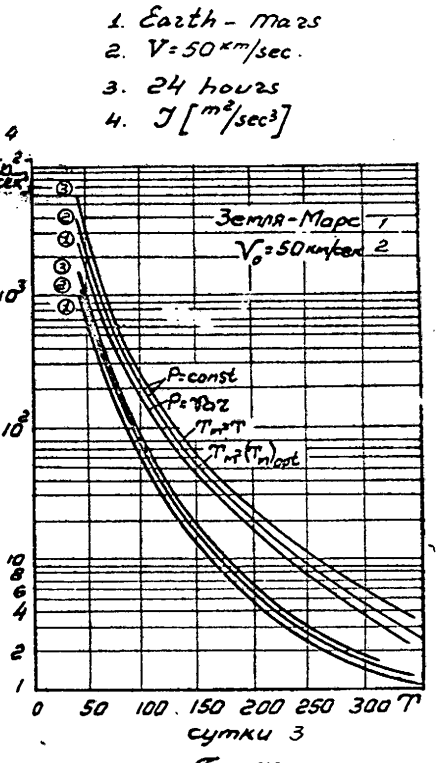


Fig. 37

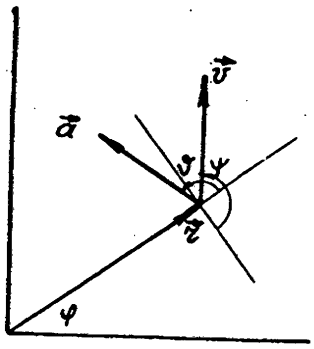
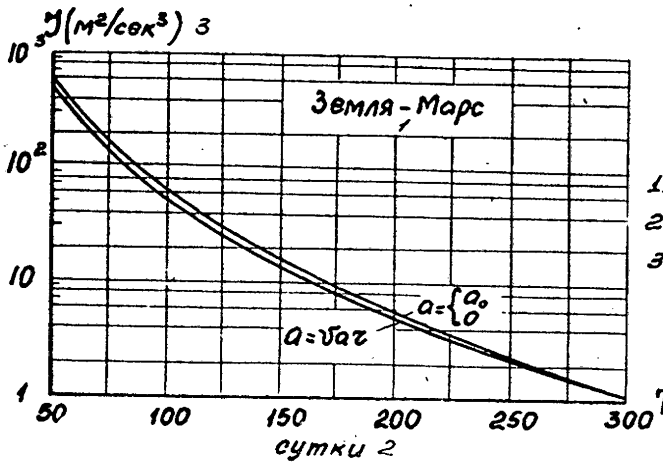
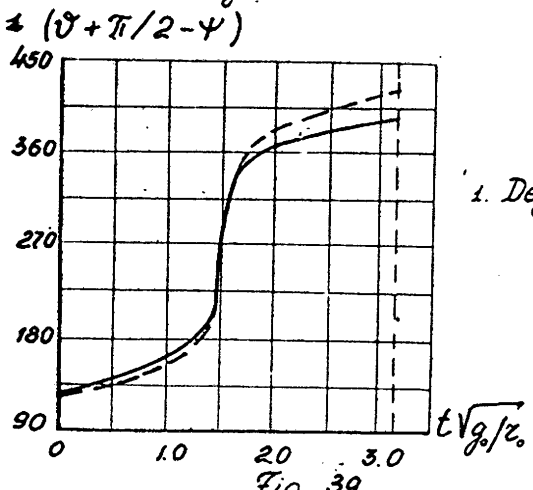


Fig. 38



1. Earth - Mars
2. 24 hours
3. $J\ [m^2/sec^3]$

Fig. 40



1. Degrees

Fig. 39

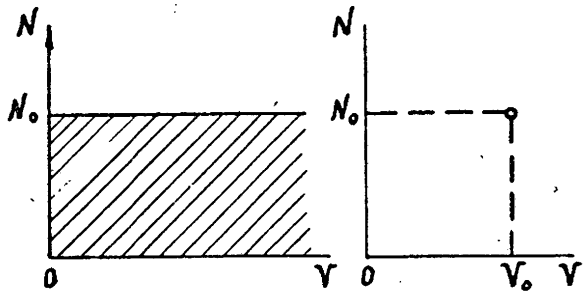


Fig. 41

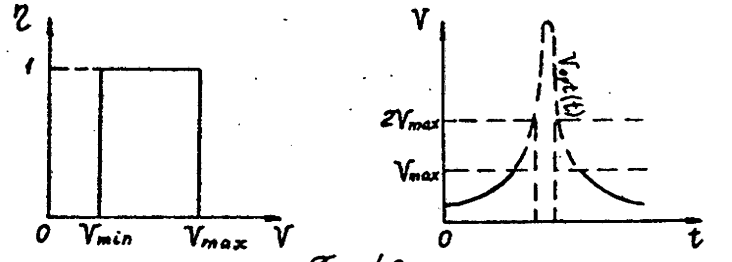


Fig. 42

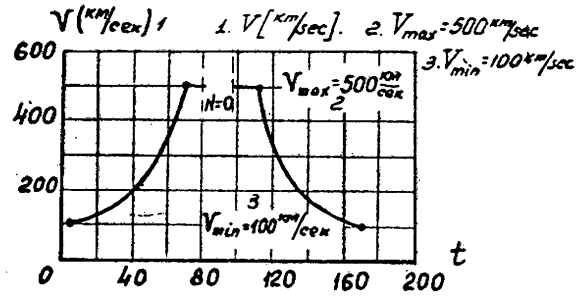
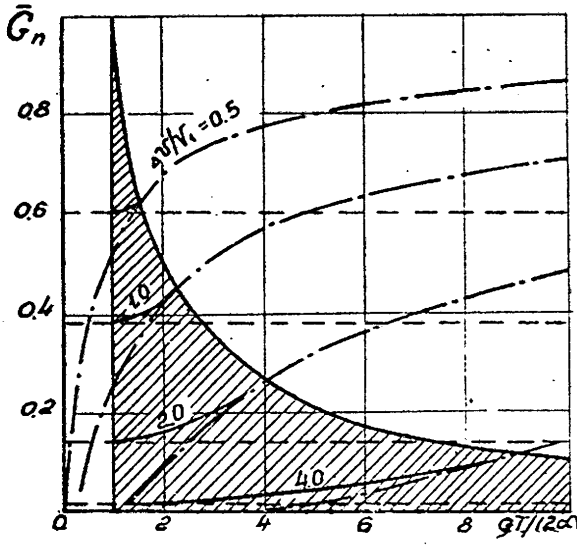


Fig. 43



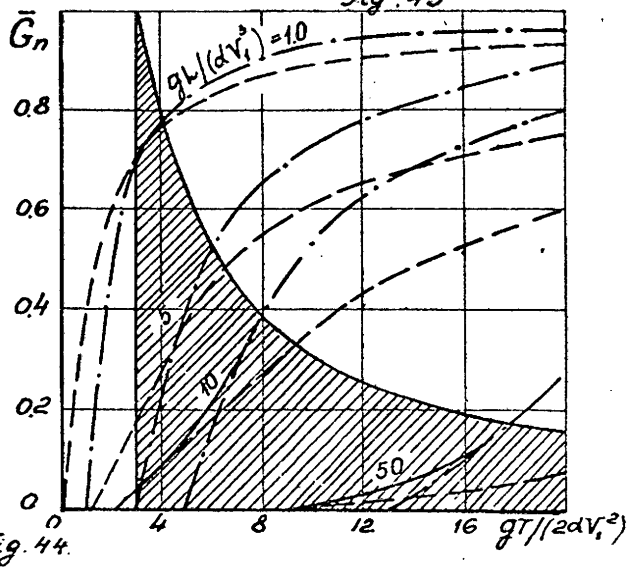


Fig. 44

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ON THE SHAPE OF HEAT REJECTING ELEMENTS COOLED BY RADIATION.

PART IV. THE EQUILIBRIUM FORMS AND SMALL VIBRATIONS OF A FLEXIBLE THREAD ROTATING IN THE FORCELESS FIELD. THE LIMITARY CHARACTERISTICS OF A BELT RADIATOR WITH MUTUAL IRRADIATION. THE OPTIMUM SYSTEM OF HEAT REJECTING FINS WITH MUTUAL IRRADIATION.

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The previous parts (I-III) of the work dealt with determining of the form of a flexible heat exchange belt (a flexible thread) in the centrifugal force field and of the optimum contour heat rejecting fins provided the emissivity is unit and the base prism surface radiant flux in negligible x/.

This paper presents the development of the investigations of the problems.

All the equilibrium forms of the rotating flexible belt are discussed and the domain of parameters within which the thread has no cross-points is outlined.

Small vibrations about the equilibrium form are also considered.

It is shown that the equation for the proper functions is that of Lame, if the thread is of small curvature and the tension is large. For the straight equilibrium form of the thread the equation reduces to the equation of Legendre and the proper functions are written in integral form.

The limitary optimum heat transfer characteristics, corresponding to the two limitary cases of radiant interference are presented.

The characteristics depend only on two dimensionless parameters and all the rest characteristics are placed between the two.

The two-dimensional variational problem of optimum contour of thin heat rejecting fins is considered, fins' mutual irradiation and the interference between the fins and the base prism surface for any emissivity being taken into account.

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x/ See: Grodzovsky G.L., Report at the XII I-AF Congress,
Grodzovsky G.L., Polov V.V., Report at the XIII IAF Congress
Grodzovsky G.L., Stasenko A.L. Report at the XIV AIF Congress.

The equilibrium forms and small vibrations of a flexible thread rotating in the forceless field

A series of works /1 - 4/ dealt with the investigation of a belt radiator.

In ref./3/, for instance, the equilibrium form of a free flexible belt, rotating in the forceless field and pressed to the surface of a cooled drum by centrifugal forces, has been defined. The main object of this paper is to investigate all the equilibrium forms of a rotating flexible thread (including a thread with cross-points), to determine the conditions under which the thread is convex and to consider the equation of small vibrations of the thread about the equilibrium form.

1. Let the thread with the length L and linear density μ rotate with a constant angular velocity $\bar{\omega}$ and simultaneously moved with a constant, tangential velocity $\bar{V}$, around one or several drums of radius r_0, which revolve with angular velocity ω_0 (fig.1). In the case of assume a slip between the drums and the thread we have

$$V = r_0(\omega_0 - \omega). \tag{1.1}$$

The thread element, having $\bar{r}$ as a radius vector with respect to the point of rotation is affected by the inertial force, equal to

$$\bar{\Phi} = \mu\omega^2\bar{r} + (2\mu V\omega + \mu V^2\varkappa)\bar{n}_0, \tag{1.2}$$

where $\varkappa$ - thread curvature, $\bar{n}_0$ - unit vector of the normal to the element. As is known, the equilibrium form of a free flexible thread is described by the equation

$$\bar{\Phi} + \frac{d}{dS}(\lambda\bar{\tau}_0) = 0, \tag{1.3}$$

where S - arc length, $\bar{\tau}_0$ - unit vector of a tangent, λ - thread tension.

Introduce non - dimensional values

$$\rho = \frac{r}{r_0}\ ;\ \sigma = \frac{S}{r_0}\ ;\ l = \frac{L}{r_0}\ ,\ \kappa = \varkappa r_0$$

$$\Lambda = \frac{\lambda}{\frac{1}{2}\mu\omega^2 r_0^2}\ ,\ a = \frac{\lambda_0}{\frac{1}{2}\mu\omega^2 r_0^2}\ ,\ b = \frac{2V}{\omega r_0}, \tag{1.4}$$

where λ_0 - the tension in the contact point of the thread with the drum; now we shall write the equation (1.2) in the projections on the axes $\bar{\tau}_0, \bar{n}_0$

$$\frac{d\Lambda}{d\sigma} + \bar{\rho}\bar{\tau}_0 = 0,\quad \bar{\rho}\bar{\tau}_0 = \frac{(\rho^2)'}{(\rho^2+\rho'^2)^{1/2}} \tag{1.5}$$

$$\bar{\rho}\bar{n}_0 + 2b + \frac{b^2}{2}\kappa - \Lambda\kappa = 0,\quad \kappa = \frac{\rho^2 + 2\rho'^2 - \rho\rho''}{(\rho^2+\rho'^2)^{3/2}} \tag{1.6}$$

After integrating (1.5) we have

$$\Lambda = \gamma - \rho^2 \qquad (1.7)$$

where γ – integration constant.

On the basis of the derived relationships the symmetry with respect to φ is seen.

In terms of new variables $\rho^2 = \xi$ (1.8)

$$\frac{\rho}{(\rho^2+\rho'^2)^{1/2}} = \eta \qquad (1.9)$$

using the equation (1.7), we can rewrite (1.6) in the form

$$\eta + B - (\gamma - \tfrac{1}{2}B^2 - \xi)\frac{d\eta}{d\xi} = 0, \quad \kappa = 2\frac{d\eta}{d\xi}. \qquad (1.10)$$

Hence

$$\eta + B = C(\gamma - \tfrac{1}{2}B^2 - \xi)^{-1} \qquad (1.11)$$

Here C – integration constant

On the ground of (1.10) we have

$$\kappa = 2C(\tfrac{1}{2} - \xi)^{-2} \qquad (1.12)$$

Thus, the thread curvature will be a continuous not decreasing with respect to a module, having a constant sign function. It is clear, that at the point of contact with the drum the thread curvature cannot be superior to the drum curvature, $k_0 \leq 1$; the sign of equality takes place only when the thread length l is equal to l_{min}.

2. In the case of considering only one drum $l_{min} = 2\pi$ and

$$a_{min} = 2(1 + \tfrac{1}{2}B)^2 \qquad (2.1)$$

On the plane ab all the threads must lie (fig.2) above the parabola (2.1). The initial conditions in the case of one drum are as follows:

$$\rho = 1, \quad \rho' = 0 \quad (\text{at the point } \varphi = \varphi_0) \qquad (2.2)$$

Then, by defining the integration constants γ and C from the equations (1.7) and (1.11), we get expressions for non-dimensional tensions and thread curvature

$$\Lambda = a + 1 - \rho^2 \qquad (2.3)$$

$$\kappa = \frac{2(1+B)(a - \tfrac{1}{2}B^2)}{(a - \tfrac{1}{2}B^2 + 1 - \rho^2)} \qquad (2.4)$$

$$\kappa_0 = \kappa(1) = \frac{2(1+B)}{a - \tfrac{1}{2}B^2} \qquad (2.5)$$

$K(1)$ as a function of the parameter b is presented in fig.3. By integrating (2.4), we shall get the equation of the thread by means of a quadrature

$$\int_1^{\rho} \frac{[(a-\frac{1}{2}b^2)+b(\rho^2-1)]\,d\rho}{\rho\sqrt{\rho^2[(a-\frac{1}{2}b^2)-(\rho^2-1)]^2-[(a-\frac{1}{2}b^2)+b(\rho^2-1)]^2}} \qquad (2.8)$$

Let R be the maximum thread radius vector, satisfying the condition

$$\rho' = 0, \quad \varphi = \pi \qquad (2.1)$$

third power polynomial with respect to ρ^2 in the radicand of the equation (2.8) has the following roots

$$\rho_1^2 = 1 \qquad (2.9)$$

$$\rho_{2,3}^2 = (a-\tfrac{1}{2}b^2) + \tfrac{1}{2}(b^2+1) \mp \tfrac{1}{2}(1+b)\sqrt{4(a-\tfrac{1}{2}b^2)+(1-b^2)} \qquad (2.9)$$

the sign before the root is chosen so that $\rho_2^2 \le \rho_3^2$. the first of them means that the thread contacts with the drum (see the initial conditions (2.2)). for all $b>-1$ in the interval $\rho_1^2<\rho^2<\rho_3^2$ the investigated third power polynomial is negative; therefore $\rho_2 = R$ and defines the largest relative radius, so that

$$1 \le \rho \le R, \quad \varphi_0 \le |\varphi| \le \pi \qquad (2.10)$$

Besides, $K(1)>0$ for $b>-1$ (fig.3) (the signs of equality are reached only at the boundary value $b=-1$). Consequently, at $b>-1$ the thread is convex, does not contain cross-points and its form is described by the integral (2.6), which is reduced to the sum of elliptical integrals[3] of the first and the third kinds $F(\psi,k)$ and $\Pi(\psi,h,k)$

$$\varphi-\varphi_0 = \frac{b}{\sqrt{\rho_3^2-1}}F(\psi,k) + \frac{a-\frac{1}{2}b^2-b}{\sqrt{\rho_3^2-1}}\Pi(\psi,h,k) \qquad (2.11)$$

$$\sin^2\psi = \frac{\rho^2-1}{\rho_3^2-1}, \quad h^2 = \rho_2^2-1, \quad k^2 = \frac{\rho_2^2-1}{\rho_3^2-1}$$

the angle φ_0 is determined by the condition $\varphi=\pi$ at $\rho=R$ and is expressed by means of full elliptical integrals

$$\pi-\varphi_0 = \frac{b}{\sqrt{\rho_3^2-1}}K(k) + \frac{a-\frac{1}{2}b^2-b}{\sqrt{\rho_3^2-1}}\Pi(h,k) \qquad (2.12)$$

the ratio

$$\tfrac{1}{2}\ell = \varphi_0 + \frac{a-(\rho_3^2-1)}{\sqrt{\rho_3^2-1}}K(k) + \sqrt{\rho_3^2-1}\,E(k) \qquad (2.13)$$

where $E(k)$ - full elliptical integral of the second kind , connects the non-dimensional initial tension a with the thread length ℓ .

At $b<-1$ the thread curvature becomes negative, and its cross-point $\rho_0, \varphi_0=\pi$ is found on the basis of the condition

$$\pi-\varphi_0 = \int_1^{\rho_0} \frac{[(a-\frac{1}{2}b^2)+b(\rho^2-1)]\,d\rho}{\rho\sqrt{\rho^2[(a-\frac{1}{2}b^2)-(\rho^2-1)]^2-[(a-\frac{1}{2}b^2)+b(\rho^2-1)]^2}} \qquad (2.14)$$

the thread form may be also expressed in terms of special functions, which are connected with elliptical integrals (for example, by Yacobi-function or by Weierstrass-function).

the dependence of the thread form on b for $l = 2,5$ is shown in fig.4.

3. Let us consider N similar drums with mass M_0, embraced by one belt and situated symmetrically about the centre of inertia O of the system. Angle $0 \le \varphi \le \pi/N$ is measured from the ray OO_1. In the region $2\psi_0$ the belt contacts the drum (fig.5). The initial conditions, of contact at the point A have the form

$$\rho = \rho_1 , \quad \rho' = -\rho_1 \operatorname{tg}(\psi_0 - \varphi_0) \qquad (\rho_1 = r_1/r_0) \quad \text{at } \varphi = \varphi_0 \tag{3.1}$$

for the point B the symmetry condition gives

$$\rho = \rho_{min} , \quad \rho' = 0 \qquad \text{at } \varphi = \pi/N \tag{3.2}$$

By integrating the equation (1.12) with regard for (1.7) and (3.1) we get the following expression for the tension

$$\Lambda = a + \rho_1^2 - \rho^2 \tag{3.3}$$

and the equation of thread in the form of the following quadrature in the domain $\varphi - \varphi_0$

$$= \int_{\rho_1}^{\rho} \frac{[(a - \tfrac{1}{2}b^2)\rho_1 \cos(\psi_0 - \varphi_0) - b(\rho_1' - \rho')]\, d\rho}{\rho\sqrt{\rho^2[(a-\tfrac{1}{2}b^2)+(\rho_1'-\rho')^2] - [(a-\tfrac{1}{2}b^2)\rho_1\cos(\psi_0-\varphi_0)+b(\rho_1'-\rho')]^2}} \tag{3.4}$$

In order to define the terms $a, \rho_1, \psi_0, \rho_0$ and P where P is equal to the ratio of OO_1, shown in fig.5, to the radius r_0, we shall make use of the following side geometrical and force conditions:
the condition (3.2) and

$$\rho_1^2 = 1 + P^2 + 2P\cos\psi_0 , \quad 1 = \rho_1^2 + P^2 - 2\rho_1 P\cos\varphi_0$$
$$l = 2N(\psi_0 + \int_{\varphi_0}^{\pi/N} d\varphi) \qquad a\sin\psi_0 = MF_1 \tag{3.5}$$
$$\left(M = \frac{M_0 P_0 \omega^2}{f\omega^2 r_0^2} \right)$$

Here P_0 – radius-vector of the point of application of the centrifugal force, acting on the drum; and the second term in the right part of the last equation (3.5) is the OO_1 projection of the forces, acting on the part of the thread, lying on the drum. Since $a < \infty$ it follows from the last equation (3.5), that ψ_0 never equals zero. Consequently, $\psi_0 - \varphi_0 > 0$, $\rho' < 0$, and the thread is entirely within the round, described about the system of drums.

It is of interest to pay attention to some particular cases (fig.6).

1) Straight thread. Physically it means, that the normal component of the centrifugal force is constant and equal to the Coriolis force, so that

$$b = -\rho_{min} \tag{3.6}$$

Here the equation of the thread (3.4) has the form

$$\rho = \frac{\rho_{min}}{\cos(\pi/N - \varphi)} \tag{3.7}$$

and geometrical parameters are defined by the ratios

$$l = 2\pi + 2NP\sin\pi/N , \quad \rho_{min} = 1 + P\cos\pi/N , \quad \psi_0 = \pi/N$$
$$\rho_1^2 = 1 + P^2 + 2P\cos\pi/N , \quad \cos(\pi/N - \varphi_0) = \rho_{min}/\rho_1 \tag{3.8}$$

2) boundary of the uniqueness of the function. The tangent to the thread at A passes through the centre of symmetry of the system

Here

$$\psi_0 - \varphi_0 = \tfrac{1}{2}\pi \qquad \sin\varphi_0 = 1/p , \tag{3.9}$$

$$\varphi - \varphi_0 = -\frac{b}{2}\int_{\rho_1}^{\rho} \frac{(\rho_1^2 - \rho^2)\, d\rho^2}{\sqrt{\rho^2[(a-\tfrac{1}{2}b^2)+(\rho_1^2-\rho^2)]^2 - b^2(\rho_1^2-\rho^2)^2}} \tag{3.10}$$

3) At further increase of ψ_0 the thread curvature becomes negative, with $0 \le \varphi_0 \le \arcsin 1/p$

$$\rho_*^2 - \rho_1^2 - \frac{\rho_1 \cos(\psi_0 - \varphi_0)(a - \tfrac{1}{2}b^2)}{b - \rho_1 \cos(\psi_0 - \varphi)} \le \rho_1^2 \tag{3.11}$$

$\rho_* \varphi_*$ - the point, where the condition $\rho' = \infty$ is fulfilled).

The case for $\varphi_0 = \arcsin 1/p$, $\rho_0 = \rho_1$ corresponds to the boundary of uniqueness, considered above. The case of $\varphi_0 = 0$ when the thread connects the radius-vector of the drum centre, is a limiting one in the sense, that with further increase of ψ_0 the thread cross-points appear and the thread cannot be considered as the best model.

4. Consider small vibrations of a flexible thread about the equilibrium form $\rho^0(\varphi)$ (all the values, characterising the equilibrium thread, will be designed by an index 0). Here radius-vector of a vibrating thread-point with a coordinate σ (nondimensional arc length) at the time moment $t = \tau/\omega$ is represented as

$$\bar{\rho}(\sigma,\tau) = \bar{\rho}^0(\sigma) + n(\sigma,\tau)\bar{n}_0^0 . \tag{4.1}$$

We shall consider the smallness of vibrations in the usual sense

$$\left|\frac{\partial n}{\partial \sigma}\right| < \varepsilon , \tag{4.2}$$

where $\varepsilon > 0$ - a small constant quantity.

Let $\bar{\tau}_0$ and $\bar{n}_0$ be the unit vectors of the tangent and of the normal to the vibrating thread. The force of inertia, acting on a unit length of the vibrating thread is equal to

$$\bar{\Phi} = \rho\omega^2 r_0 \left\{\bar{\rho} + \left[\left(\frac{b}{2}\right)^2 k + b\right]\bar{n}_0 - \frac{\partial^2 n}{\partial\tau^2}\bar{n}_0^0 + 2\frac{\partial n}{\partial\tau}\bar{\tau}_0^0\right\}. \tag{4.3}$$

In this equation the last two terms denote a non-dimensional inertial force, corresponding to acceleration $(\partial^2 n/\partial\tau^2)\bar{n}_0^0$, and to the Coriolis force, resulting from the motion in the rotating system with a relative velocity $(\partial n/\partial\tau)\bar{n}_0$. Represent the tension in the form

$$\Lambda(\sigma,\tau) = \Lambda^0(\sigma) + \Lambda^1(\sigma,\tau) , \tag{4.4}$$

where Λ^1 — perturbation.

By writing (1.3) in the projections on the axis $\bar{\tau}_0^0$ and $\bar{n}_0^0$, we get two systems : an undisturbed one (1.5), (1.6) and a system, describing small vibrations

$$\frac{1}{2}\frac{\partial \Lambda^1}{\partial\sigma} + \left[\frac{1}{2}\left(\Lambda - \frac{1}{2}b^2\right)k - b\right]\frac{\partial n}{\partial\sigma} + 2\frac{\partial n}{\partial\tau} = 0 \tag{4.5}$$

$$n - \frac{\partial^2 n}{\partial\tau^2} + \frac{1}{2}\frac{\partial}{\partial\sigma}\left(\Lambda\frac{\partial n}{\partial\sigma}\right) - \frac{1}{2}(\Lambda - \Lambda^0)k^0 + \left(\frac{b}{2}\right)^2(k - k^0) = 0 \tag{4.6}$$

The set of the equations (4.6) and (4.5) is nonlinear. We linearize it assuming the curvature to be small $|K^0(\sigma)| < \varepsilon$ and the solution to be rather large $\Lambda^0 \gg |\Lambda^1|$. These conditions may take place in the case when the belt moves around two or more heavy drums. Then the equation (4.6) is written in the form

$$\eta - \frac{\partial^2 \eta}{\partial \tau^2} + \frac{1}{\varepsilon} \frac{\partial}{\partial \sigma} \left[(\Lambda^0 - \tfrac{1}{2} b^2) \cdot \frac{\partial \eta}{\partial \sigma} \right] = 0 \qquad (4.7)$$

and will not depend on Λ^1. Having solved it we may find Λ^1 from (4.5) with the help of the quadrature.

We shall search the solution to (4.7) in the form $h(\sigma, \tau) = S(\sigma) T(\tau)$

Then for $T_\nu(\tau)$ we shall find

$$T_\nu = C_{1\nu} \exp(i\sqrt{\lambda_\nu - 1}\,\tau) + C_{2\nu} \exp(-i\sqrt{\lambda_\nu - 1}\,\tau) \qquad (4.8)$$

and for definition of $S(\sigma)$ we shall find the equation

$$\frac{d}{d\sigma} \left[(\Lambda^0 - \tfrac{1}{2} b^2) \frac{dS}{d\sigma} \right] + \nu S' = 0 \qquad (4.9)$$

With the help of the ratio $x = F(\psi, k)$

$$x = \int \frac{d\rho^2 \sqrt{\zeta_3^2 - \zeta_1^2}}{2\sqrt{\rho^2 [(a - \tfrac{1}{2} b^2) + (\rho_1^2 - \rho^2)]^2 - [(a - \tfrac{1}{2} b^2)\rho_1 \cos(\psi_0 - \psi_0) - b(\rho_1^2 - \rho^2)]^2}} \qquad (4.10)$$

$$\sin^2 \psi = \frac{\rho^2 - \zeta_1^2}{\zeta_3^2 - \zeta_1^2}, \quad k = \frac{\zeta_2^2 - \zeta_1^2}{\zeta_3^2 - \zeta_1^2}$$

we introduce a new variable.

Here ζ_1^2, ζ_2^2, ζ_3^2 - the roots of the denominator in the ratio (4.10).

Then the equation (4.9) is in the form

$$\frac{d^2 S}{dx^2} = \nu \left[k^2 \operatorname{Sn}^2 x - \frac{a - \tfrac{1}{2} b^2 - \rho_1^2 - \zeta_1^2}{\zeta_3^2 - \zeta_1^2} \right] S \qquad (4.11)$$

In the particular case of $b = -\rho_{min}$ we get the following equation of Legendre.

$$\frac{d}{d\sigma_1} \left[(1 - \sigma_1^2) \cdot \frac{dS}{d\sigma_1} \right] + \nu S = 0 \qquad \sigma_1 = \frac{\sqrt{\rho_1^2 - \rho_{min}^2} - \sigma}{\sqrt{a - \tfrac{1}{2} b^2 + \rho_1^2 - \rho_{min}^2}} \qquad (4.12)$$

The equations (4.11) and (4.12) with $\nu = m(m+1)$, $m > 0$ integer) have the solutions, called Lame functions /3/ and Legendre polynomial, respectively. In particular, the equation of Lame was considered for m too, when is equal to an integer plus one half /3/. As in our case, the proper values themselves must be determined, the equation /4.11/ was been being solved numerically. In fig.1 several first proper values for the problem with uniform boundary conditions are represented

The equation (4.12) in the general case has the following fundamental solutions /1/:

$$S_{1\nu}(\alpha) = 4 \int_0^{\alpha + \pi/2} \frac{\cos\sqrt{\nu + \lambda}\, z}{\sqrt{\cos z + \sin\alpha}}\, dz = S_{2\nu}(-\alpha) \qquad (4.13)$$

$$\alpha = \arcsin\sigma.$$

We shall build the proper functions in the form of linear combinations of these fundamental solutions

$$S_\nu = A_\nu S_{1\nu} + B_\nu S_{2\nu} \qquad (4.14)$$

and proper values will be found from the equation

$$\begin{vmatrix} S_{1\nu}(\alpha_1) & S_{2\nu}(\alpha_1) \\ S_{1\nu}(-\alpha_1) & S_{2\nu}(-\alpha_1) \end{vmatrix} = 0 \qquad (4.15)$$

$$\alpha_1 = \arcsin \frac{\sqrt{\beta_1^2 - \beta_{min}^2}}{\sqrt{a - \frac{1}{2}\delta^2 + \beta_1^2 - \beta_{min}^2}}$$

the constant B_ν is equal to

$$B_\nu = - \frac{S_{1\nu}(\alpha_1)}{S_{2\nu}(\alpha_1)}$$

the constant A_ν is determined from the normalization condition

$$\int_{-\alpha_1}^{\alpha_1} S_\nu^2 \, d\alpha = 1$$

B. Limiting characteristics of a belt radiator with mutual irradiation

In refs. /1,2,4/, which dealt with the investigation of a belt radiator, the effect of the belt mutual irradiation on the radiator characteristics was not considered. It was only mentioned, that if it would be made into a double strip, the mutual irradiation would be decreased; an attempt was made to use "the effective belt width" in the temperature distribution with length $T^0(S)$, got in the case when mutual irradiation was neglected. If it is taken into account the form of the very function $T(S,Z)$, determined by integro-differential equation with a kernel, depending on the belt form and, consequently, on many concrete parameters, is changed. However, the boundaries, depending only on two parameters, of the domain, in the limits of which the belt radiator characteristics (such as temperature distribution, mass and typical dimensions) at any belt form will be situated, may be marked.

The determination of this domain is the main object of the paper.

Let the belt with the length L move over the drum, with the radius r_0, which is being cooled; by contacting it the belt is being heated in the interval of temperatures from T_1 to $T_2 > T_1$. Assume that:

1) the belt is physically thin, $\delta/\lambda \to 0$ (δ - belt thickness, λ - thermal conductivity);

2) $r_0 \ll L$ then the temperature distribution equation and the equation of the accumulated and radiated energy balance may be given in the form

$$c\delta\gamma_0 \frac{\partial T}{\partial \beta} = -2\varepsilon\sigma_0 T^4(S,Z) + \Phi(S,Z) = -q_1(S,Z) \qquad (1)$$

$$\Phi(S,Z) = \int_0^L \int_{-\beta_0/2}^{\beta_0/2} H(S_0,Z_0)\, K(S,Z;S_0,Z_0)\, dS_0\, dZ_0 \qquad (2)$$

$$Q = M_1 c \frac{V}{L} \cdot \frac{1}{\beta} \int_{-\beta_0/2}^{\beta_0/2} [T_2(z) - T_1(z)]\, dz \qquad Q = \int_0^L \int_{-\beta_0/2}^{\beta_0/2} q(S,Z)\, dS\, dZ \qquad (3)$$

Here ρ, C - density and belt specific heat,

δ, B, M_L - thickness, width and mass of the belt,

V - linear velocity,

ε - surface emissivity,

σ_0 - Stefan - Boltzmann constant,

H - effective radiation flow,

K - the *kernel*, depending on the belt form,

Q - heat, conducted from the surface per unit time.

The two addends in the right half of the equation (1) describe radiant flows from the outer and inner (with mutual irradiation) surfaces of the belt it may be shown, that

$$2\varepsilon \langle q_1/\sigma_0 \rangle > T_2^4 \geq \langle \varepsilon\tau^4(s) - \frac{1}{2/\varepsilon - 1}[\tau^4(1-s) - \tau^4(s)] \rangle = \varepsilon \langle \tau^4 \rangle \qquad (4)$$

$$\tau = \frac{T}{T_2}, \quad s = \frac{S}{L}, \quad \langle f \rangle = \int_0^1 f(s)\,ds$$

The left half of inequality (4) corresponds to the belt without mutual irradiation, the right one corresponds to a closed flat belt of any finite width (fig.6), with the elements moving parallel and towards each other at an infinitely close distance (for one and two drums temperature distribution is evidently similar). In these two limitary cases the temperature distribution across the belt is described by the following equations

$$\alpha \frac{d\tau}{ds} = -\varepsilon [\tau^0(s)]^4 \qquad (5)$$

$$\alpha \frac{d\tau}{ds} = -\varepsilon\left(1 + \frac{1}{2-\varepsilon}\right)\tau^4(s) + \frac{\varepsilon}{2-\varepsilon}\tau^4(1-\varepsilon)$$

$$\alpha = \frac{\rho c \delta V}{\sigma_0 T_2^3 L} \qquad (6)$$

The initial condition is

$$\tau^0(0) = \tau(0) = 1 \qquad (7)$$

The equation (5) is being integrated in the final form /1/ and determines oneparametrical family of curves.

$$\tau^0(S, \alpha/\varepsilon) = \left(1 + \frac{6\varepsilon}{\alpha} S\right)^{-1/3} \qquad (8)$$

The equation (6) was solved numerically with the help of the iteration method, with the function $\tau_{n-1}(s) = k[\tau_{n-1}(s) - \tau_{n-2}(s)]$ substituted into the right half of the qualion for determining $\frac{\partial \tau_n}{\partial S}$. Fig.9 gives the example of a convergence process depending on k. Let $\delta_{n,n-1} = \left[\int_0^1 (\tau_n - \tau_{n-1})^2 ds\right]^{1/2}$ be root-mean- square error. The process was finished under the condition $\delta_{N,N-1} \leq \omega$ (N - a full number of assumptions, ω - reference accuracy)

For calculations it was chosen:

$$\tau_{-1} \equiv \tau_0 \equiv \tau^0(s), \quad k = 1/4, \quad \omega = 10^{-4}, \quad \Delta S = 10^{-2}, \quad 10^{-1} \leq \alpha/2 \leq 10, \quad \frac{1}{2}\Delta\alpha = 10^{-1},$$

$$10^{-1} \leq \varepsilon \leq 1, \quad \Delta\varepsilon = 10^{-1}$$

In fig.10 a two-parametrical family of the curves $\tau(S;\alpha;\varepsilon)$ is represented. The qualitative difference between the functions $\tau^0(S;\alpha/\varepsilon)$ and $\tau(S,\alpha,\varepsilon)$ is in that the derivative of the last function changes its sign, as to the case of $S=1$ the retaining belt is heated again; it is caused by a positive total radiation flux. Let us introduce the following non-dimensional terms:

$$m=\frac{M_1}{Q^2/B\sigma_0 c\nu T_2^5}\;;\qquad \ell=\frac{L}{Q/B\sigma_0 T_2^4} \tag{9}$$

$$\Delta=\frac{\delta}{Q/B\rho c\nu T_2}\;,\qquad \tau_1=\tau(1)=\frac{T_1}{T_2}$$

Then from the equations (2) and (3) we shall have

$$m=\Delta\ell \qquad \ell=\langle q_1\rangle^{-1}$$

$$\langle q\rangle=\frac{\langle q_1\rangle}{\sigma_0 T_2^4}\;,\quad \Delta=(1-\tau_1)^{-1}. \tag{10}$$

From (4) it follows

$$\langle q\rangle=\varepsilon\langle\tau^4\rangle,\quad \langle q^0\rangle=2\varepsilon\langle(\tau^0)^4\rangle$$

If, with the help of (6), we neglect mutual irradiation, we have

$$m^0=\left(\frac{1}{\tau_1^{0\,3}}-1\right)\Big/\sigma\varepsilon(1-\tau_1^0)^2$$

$$\ell^0=\left(\frac{1}{\tau_1^{0\,3}}-1\right)\Big/\sigma\varepsilon(1-\tau_1^0) \tag{11}$$

The functions $\tau_1(\alpha,\varepsilon)$, $\ell(\alpha,\varepsilon)$, $\Delta(\alpha,\varepsilon)$, $m(\alpha,\varepsilon)$ (non-dimensional mass of the belt) $\mu(\alpha,\varepsilon)=m/m^0$ and $\gamma(\alpha,\varepsilon)=\langle q\rangle$ (coefficient of mutual irradiation) are shown in fig. 11-15. It is seen, that there exists min m when respect to $\alpha(\varepsilon)$

As $\tau_1=\tau_1(\alpha,\varepsilon)$, the optimum $\tilde{\tau}_1(\varepsilon)$, ensuring min m, exists. The optimum limiting values, characterizing "the very best" and "best of the worst" belts are given in figs.16-18. Here it is seen, that the ratio $\tilde{\tau}_1^0(\varepsilon)$ is represented by a straight line $\tau_1^0\approx 0{,}69$, because of the fact, that the solution of the equation (3) contains only one parameter $\alpha/2\varepsilon$ (the corresponding optimum value is $\tilde{\alpha}/2\varepsilon=1{,}47$). We may get this value /4/ by differentiating (11) with respect to τ_1^0; as a result we have the fourth power equation

$$(\tau_1^0)^4-5/2\,\tau_1^0-3/2=0 \tag{12}$$

in which one of the roots, satisfying to a physically clear requirement $0<\tau_1^0<1$ is equal to $\tilde{\tau}_1^0\approx 0{,}69$

At $0.1\le\varepsilon\le 1$ the ratio of the optimum limiting masses is monotonously changed from 2.00 to 2.51. It can be shown, that "one-sided" belt, radiating only from the outer surface /4/, has the mass and the length, that is exactly twice the length in the case of neglecting mutual irradiation; consequently, it lies in the limits of the domain of limiting characteristics, that was been already built. The temperature distribution is simply connected with τ^0:

$$\tau^4(S,\alpha/\varepsilon)=\tau^0(S,2\alpha/\varepsilon)$$

Optimum system of heat rejecting fins

The consideration of the questions of cooling a vehicle in space vacuum leads to the problem of finding optimum shapes of the heat irradiating surfaces. In recent years such problems have been attracting the greater and greater attention.

The simplest problem of cooling a heat rejecting thin plate of rectangular cross-section was considered in refs. /5,9/. In these papers the efficiency of flat heat rejecting fins, combined with tube radiators was investigated.

The optimum(in regard to the minimum cross-section area) form of a single heat rejecting triangular fin, uniformly heated from one end, was found in paper /10/. At the same time the problem of the optimum contour of a single heat rejecting fin in the family of curves expressed by the power law /11/, was also considered.

In papers /12-15/ the variational problem of the optimum profile of a single fin was considered and solved. In /13/ a fin with a trapezoidal profile was considered.

Papers /6-15/ dealt with a single fin. The problem of cooling a heat radiating cylinder by means of several fins (more than two) leads to the necessity of taking into account the fins mutual irradiation; in this case the temperature distribution across the fin width is found as the solution of a non-linear integro - differential equation. In papers /16-17/ the problem of the optimum number and contours of thin rectangular radiating fins was investigated. A more general problem, concerning the optimum contours and number of thin trapezoidal fins, was solved in ref./18/.

A little bit earlier, in paper /19/ the variational problem concerning the optimum profile and number of fins for radiating surfaces of unit emissivity was solved.

All the mentioned problems were considered as two-dimensional. In the problems of /10 -19/ the base cylinder or prism surface was assumed to be small in comparison with the fin radiating surface; therefore, the base surface radiation and the interference of the surface with the fins could be neglected.

In this paper the two-dimensional variational prolem of the optimum profile of heat rejecting fins with base prism interaction for different surfaces emissivities, is considered. Earlier, in /19/, this problem was considered only for the particular case of cooling a heat rejecting thread, i.e. for $X_0=0$ (fig.20), by black radiating surfaces of the fins.

In paper /20/ the optimum contour and number of thin triangular fins problem was considered, the interference of the fins with the base surface of the cooled regular prism for different surfaces' emissivities was taken into account.

Let us consider the two-dimensional problem of cooling a regular prism, radiating prescribed heat flow W from the prism unit length and having a prescribed constant surface temperature T_0. As is usual, the cooling fins

are assumed to be thin and slope, in the sense that the transversal heat flow in the fin is negligible as compared to the longitudinal one, and that each fin does not irradiate itself. [8-18]

With these assumptions in view, the thermal conductivity equation and the irradiation heat rejection law are:

$$Q(x) = -2\lambda y(x)\frac{dT}{dx} \tag{1}$$

$$dQ(x) = -2q(x)dx \tag{2}$$

where $Q(x)$ - the heat flow passing through the fin cross-section at the point x,

$y(x)$ - one half of the fin thickness,

$q(x)$ - resultant radiation from a fin side surface.

λ - fin material thermal conductivity.

For $\varepsilon < 1$ $q(x)$ is found as the solution of the following integral equation

$$q(x) = q_0(x) + (1-\varepsilon)\int_{F_\xi} q(\xi)\, G(x\xi)\, dF_\xi \tag{3}$$

where

$$q_0(x) = \varepsilon\sigma T_x^4 - \varepsilon\int_{F_x}\varepsilon\sigma T_x^4 K(xz)dF_z - \int_{F_\xi}\varepsilon\sigma T^4(\xi)K(x\xi)dF_\xi - \int_{F_z}(1-\varepsilon)\left[\int_{F_\xi}\varepsilon\sigma T^4(\xi)\left[K_1(z\xi)+K_2(z\xi)\right]dF_\xi\right]K(xz)dF_z$$

$$G(x\xi) = K(x\xi) + (1-\varepsilon)\int_{F_z}\left[K_1(z\xi)+K_2(z\xi)\right]K(xz)dF_z$$

ε - surface emissivity ;

σ - Stefan - Boltzmann constant;

$K(x\xi) = \frac{\cos\theta_1\cdot\cos\theta_2}{\pi\, r^2(x\xi)}$ - the well known kernel of integral equations in radiation problems (see fig.20).

The integrals in (3) are taken with respect to the adjacent fin and prism surfaces, "seen" from the point x .

The following variational problem should be solved: determine the cross-section profile $y(x)$ and the number n of fins, having the minimum total weight, if

x_0 - prescribed prism size,

T_0 - prism surface temperature,

W - total heat flow to be removed.

This is the problem of minimizing the functional

$$F_x = 2n\int_{x_0}^{x_1} y(x)dx \tag{4}$$

with the ties (1), (2), (3) and under the isoperimetrical condition

$$W = n(Q_0 + Q') \tag{5}$$

where Q_0 and Q' - heat flows, radiated by one fin and one prism surface respectively. We have:

$$Q' = \int_{-x_0\sin\varphi_2}^{x_0\sin\varphi_2}\left[\varepsilon\sigma T_z^4 - \int_{F_\xi}\varepsilon\sigma T^4(\xi)K(z\xi)dF_\xi + \int_{F_\xi}(1-\varepsilon)q(\xi)K(z\xi)dF_\xi\right]dz \tag{6}$$

The solution of the equation (3) may be written in the form

$$q(x) = q_0(x) + (1-\varepsilon)\int_{F_\xi}\Gamma(x\xi)q_0(\xi)dF_\xi \tag{7}$$

where $\Gamma(x\xi) = \sum_{j=0}^{\infty} (1-\varepsilon)^j G_j(x\xi)$ – the resolvent of the equation (5)

$$G_0(x\xi) \equiv G(x\xi); \quad G_j(x\xi) = \int_{F_\xi} G_0(x\xi)\, G_{j-1}(\xi\xi)\, dF_\xi$$

Substituting $q_1(x)$ from (1) into (2) and integrating the latter we receive

$$Q(x) = Q_0 - 2\int_{x_0}^{x}\left[q_0(x) + (1-\varepsilon)\int_{F_\xi}\Gamma(x\xi)\, q_0(\xi)\, dF_\xi\right] dx \qquad (8)$$

Using (5), (8) and (4) we shall find the expression for $y(x)$:

$$y(x) = \frac{\int_{x_0}^{x}\left[q_0(x) + (1-\varepsilon)\int_{F_\xi}\Gamma(x\xi)\, q_0(\xi)\, dF_\xi\right] dx - \frac{1}{2}\left(\frac{W}{n} - Q'\right)}{\lambda T'(x)} \qquad (9)$$

thus the (1), (2), (3) and the isoperimetrical condition (3) have been satisfied. The problem reduces to minimizing the functional

$$F_x = 2n\int_{x_0}^{x_1} \frac{\int_{x_0}^{x}\left[q_0(x) + (1-\varepsilon)\int_{F_\xi}\Gamma(x\xi)\, q_0(\xi)\, dF_\xi\right] dx - \frac{1}{2}\left(\frac{W}{n} - Q'\right)}{\lambda T'(x)}\, dx \qquad (10)$$

depending only on $T(x)$; here we have in mind, that after substituting $q_1(x)$ from (7) into (8) the functional Q' also depends only on $T(x)$.

Later on it will be convenient to use the following non-dimensional variables:

$$\bar{Q} = \frac{Q}{W}; \quad \bar{Q}' = \frac{Q'}{W}; \quad \bar{T} = \frac{T}{T_0}; \quad \bar{x} = \frac{x}{x^*}; \quad \bar{\xi} = \frac{\xi}{x^*}; \quad \bar{y} = \frac{y}{y^*}$$

$$\bar{q} = \frac{q}{\varepsilon\sigma T_0^4}; \quad \bar{F} = \frac{F_x}{F^*}$$

where

$$W = \frac{W}{n}; \quad x^* = \frac{W}{\varepsilon\sigma T_0^4}; \quad y^* = \frac{W x^*}{\lambda T_0} \qquad F^* = \frac{2W^3}{\lambda\sigma^2 T_0^5}$$

In terms of the new variables the functional (10) can be rewritten in the form:

$$\bar{F} = \frac{n}{\varepsilon^2}\int_{\bar{x}_0}^{\bar{x}_1} \bar{y}\left[\bar{T}(\bar{x}), \bar{x}, n, \varepsilon\right] d\bar{x} \qquad (11)$$

The unknown extremal $\bar{T}(\bar{x})$ must satisfy the following boundary conditions

$$\bar{T}(\bar{x}_0) = 1 \qquad \bar{T}(\bar{x}_1) = \bar{T}_1 \qquad (12)$$

The optimum value $\bar{x}_1$ (as the rim width is not prescribed) will be found from the equation

$$n\int_{\bar{x}_0}^{\bar{x}_1}\left[\bar{q}_0(\bar{x}) + (1-\varepsilon)\int_{F_\xi}\Gamma(\bar{x}\bar{\xi})\,\bar{q}_0(\bar{\xi})\, dF_\xi\right] d\bar{x} - \frac{1}{2}(1-\bar{Q}') = 0 \qquad (13)$$

The condition (13) meets the natural physical requirement $\bar{Q}(\bar{x}_1) = 0$

The solution to the problem (11–13) has been fulfilled with the help of an electronic computer.

The numerical solution to the problem was carried out by the following relaxation method: the root $\bar{x}_1^{(0)}$ of the equation (13) was found on the basis of an arbitrary initial approximation $\bar{T}^{(0)}(\bar{x})$, satisfying the condition (12), then the first improved approximation $\bar{T}^{(1)}(\bar{x})$ to the extremal is found by means of the steepest descent method /21,22/. The described procedure is used until

two suosequent approximations $\bar{T}^{(i)}_{(\bar{x})}$, $\bar{x}_1^{(i)}$ and $\bar{T}^{(i+1)}_{(\bar{x})}$, $\bar{x}_1^{(i+1)}$ differ by less than a prescribed magnitude.

In Figs. 21÷ 30 the examples of calculated curves for different values of $\varepsilon, \bar{x}_0, n, \bar{T}_1$ are presented. The dependence of the optimum fin profile contour on the size $\bar{x}_0$ of a cooled cylindrical prism with $\varepsilon = 0{,}5$ and $\bar{T}_1 = 0{,}5$ is plotted in Fig.21. Fig.22 presents the dependence of the fin profile on the number of fins with fixed magnitudes of the other parameters. As is seen in Fig.27 the quantity $\bar{T}_1$ begins to affect the total weight of the system of fins only from the values $0{,}65 \div 0{,}75$. As to the optimum fin profile and especially its width $\bar{x}_1 - \bar{x}_0$, the influence has been revealed earlier (Fig.23). The temperature distribution across the fin width is shown in Fig.24.

In general, the more the number of fins, the nearer the function $\bar{T}(\bar{x})$ is to the linear one (if $\bar{T}_1$ is not too small). For the systems of fins with $n=3$ the temperature distribution is close to the curve $\bar{T}(\bar{x}) = \sqrt{\frac{\bar{x}-\bar{x}_0}{\bar{x}_1-\bar{x}_0}}$, corresponding to an optimum single fin. In figures 25 and 26 the dependence of the curve $\bar{q}(\bar{x})$ on the number of fins and on the cooled prism size $\bar{x}_0$ is shown. The correlations between $\bar{F}$ and the number of heat rejecting fins for different ε and the sizes $\bar{x}_0$ of the cooled prism are represented in Figs. 28÷30. The continuous curves reflect the results of this paper. The dotted line corresponds to the most profitable fins of a triangular profile; it is built on the basis of the results of paper /10/. The dot-dash line /10/ corresponds to the most profitable fins of a rectangular profile. The dots on this figure correspond to the results of paper /11/. The correlation $\bar{F}(n) = \frac{1}{n^2}$ meeting the case of negligible fins' mutual irradiation is also plotted on the graph.

To evaluate the applicability of the produced results to real systems it should be noted, that the characteristic linear dimensions of the problem, i.e the dimensions within which the phenomenon develops, may be found from the relations

$$W \approx n x^{**} \varepsilon \sigma T_0^4 \quad \text{and} \quad W \approx y^{**} \lambda n \frac{T_0}{x^{**}}$$

Nondimensional variables, introduced by the correlations $\bar{y} y^{**} = y$, $\bar{x} x^{**} = x$ must be of the same order (of the order 1, as it is seen in the graphs, taking into account, that $x^* = n x^{**}$ and $y^* = n^2 y^{**}$)

wherefrom it follows

$$\frac{y_0}{\Delta x} \approx \frac{y^{**}}{x^{**}} = \frac{W}{n \lambda T_0}$$

where $\Delta x = x_1 - x_0$

A fin will be thin, if $\frac{y_0}{\Delta x} \ll 1$, i.e. $W \ll n \lambda T_0$. It is seen that for the prescribed W and T_0 the validity of the obtained results increases with the increase of the number of fins.

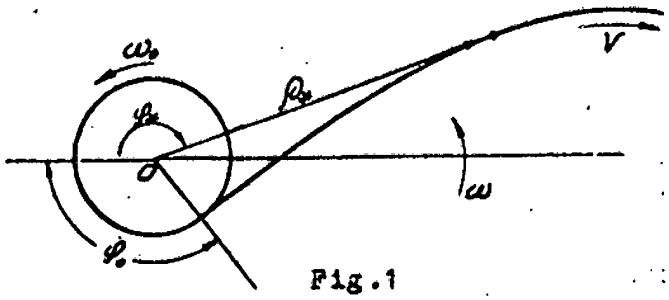


Fig.1

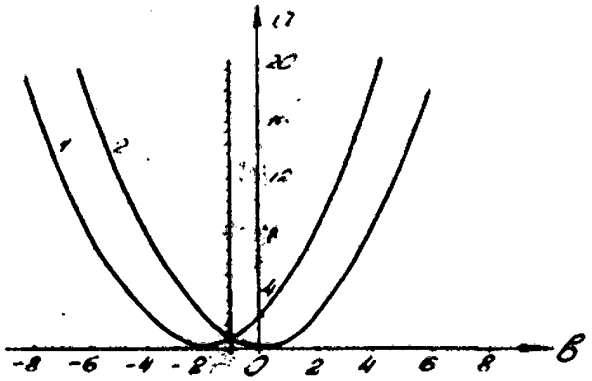


Fig. 2 1) $a=2(1+\frac{\beta}{2})$
2) $a=\frac{\beta^2}{2}$

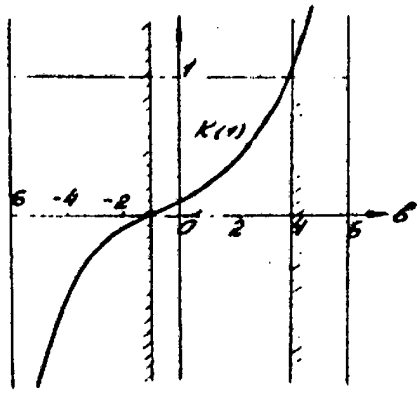


Fig.3 $a=18$

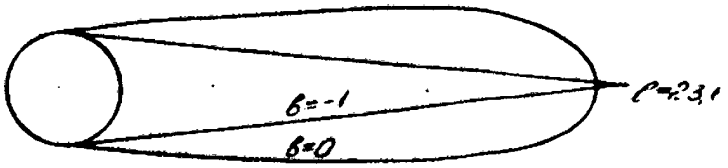


Fig.4

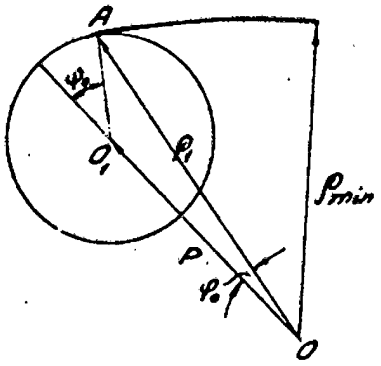


Fig.5

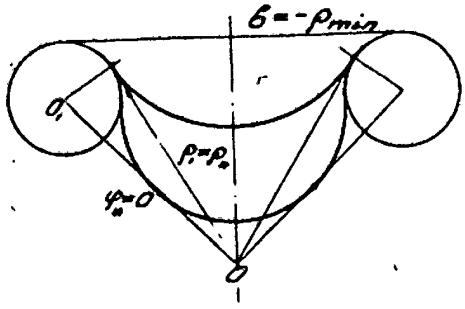
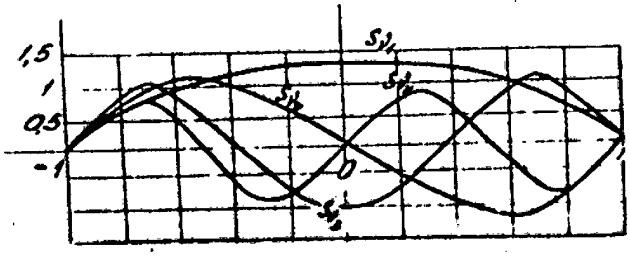


Fig.6



a=56
B=3
$\rho_1=10$
$\rho_{min}=2$

Fig.7

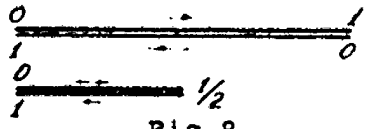


Fig.8

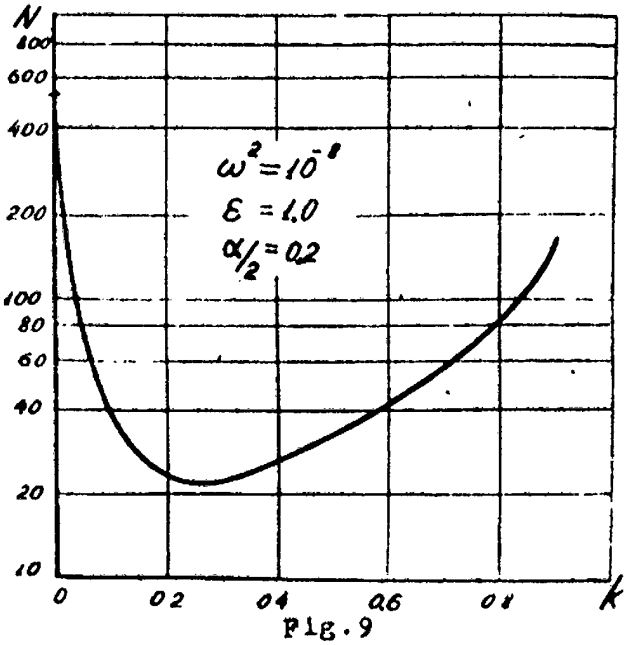


Fig.9

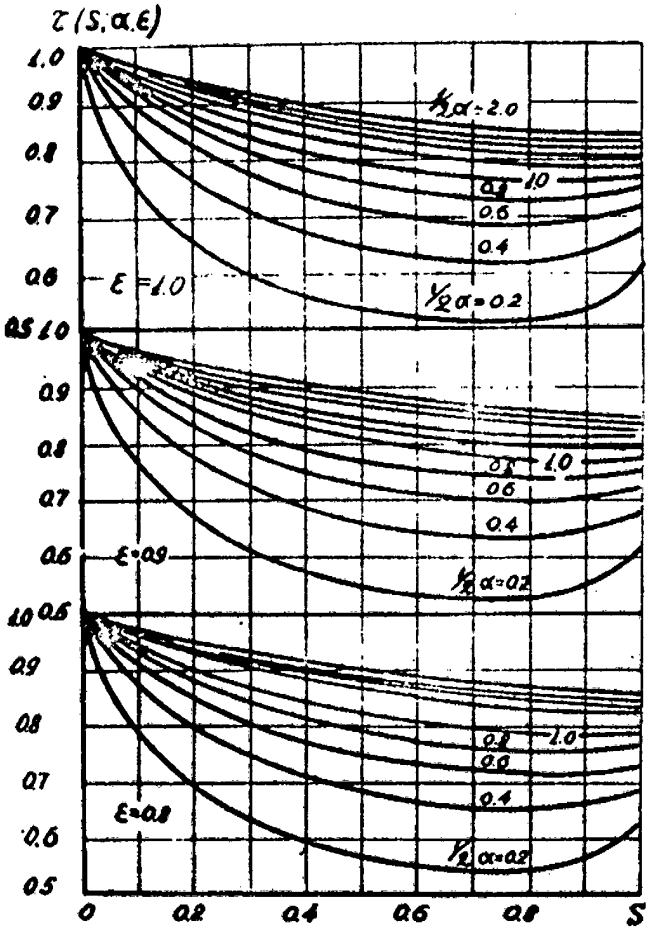


Fig.10

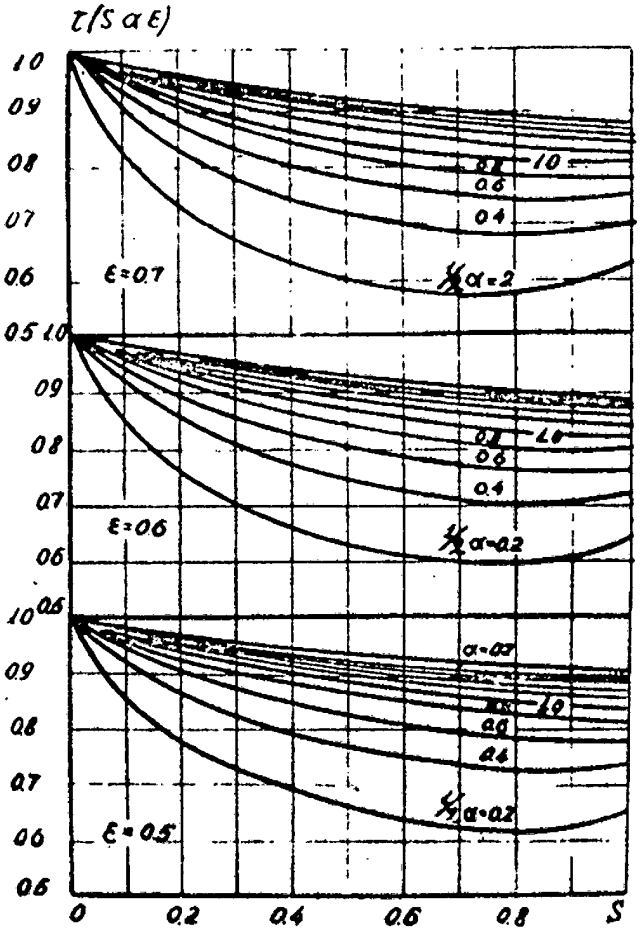


Fig.11

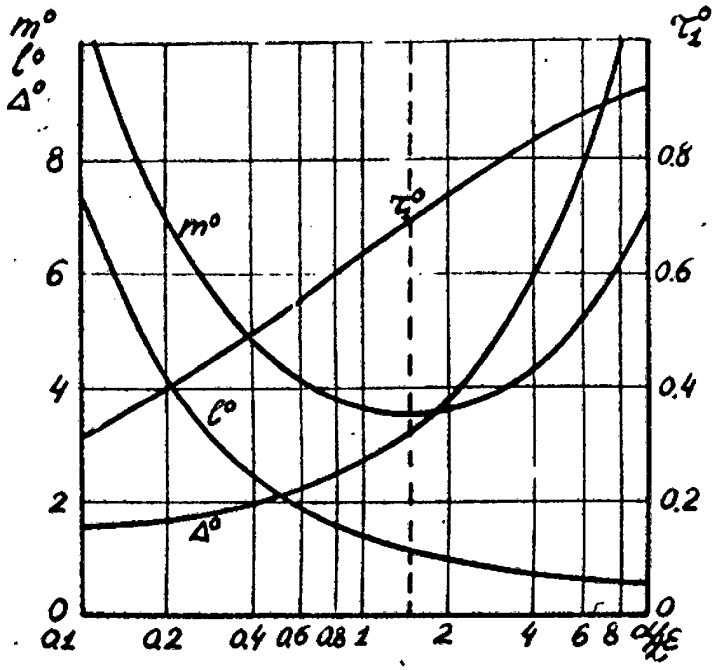


Fig.12

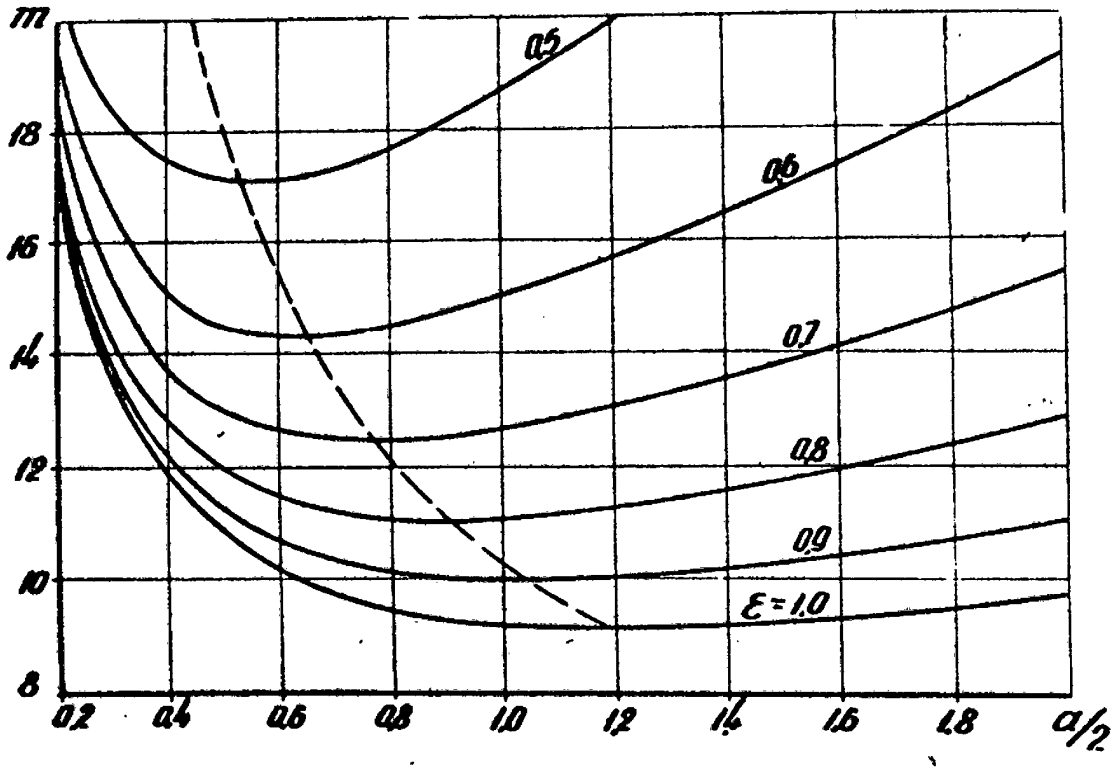


Fig.13

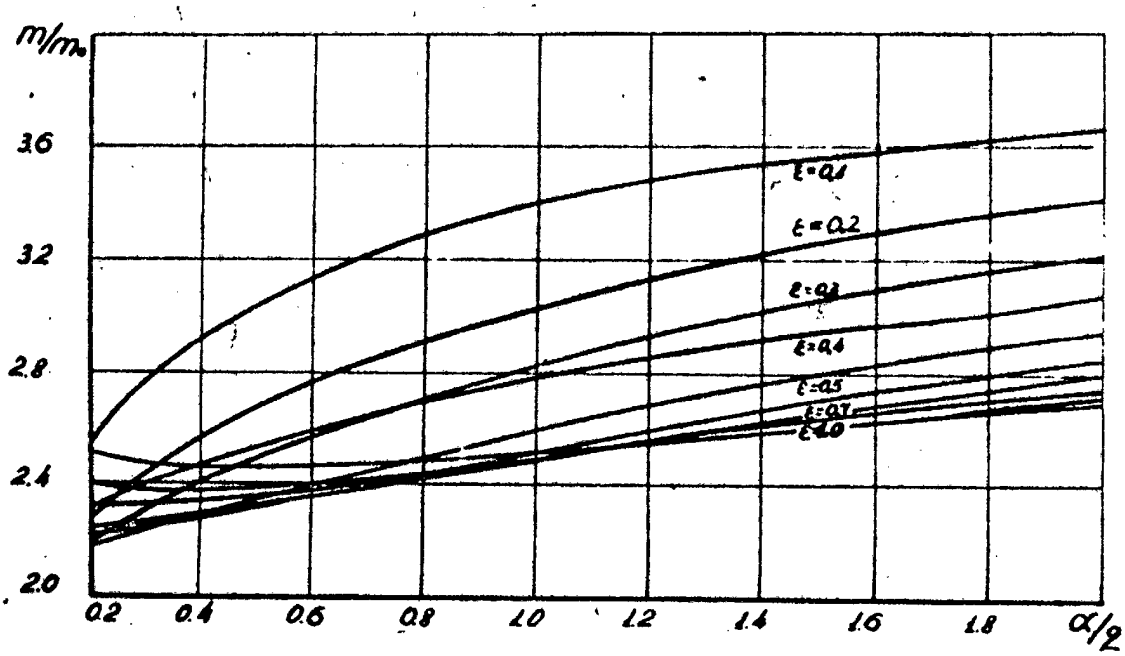


Fig. 14

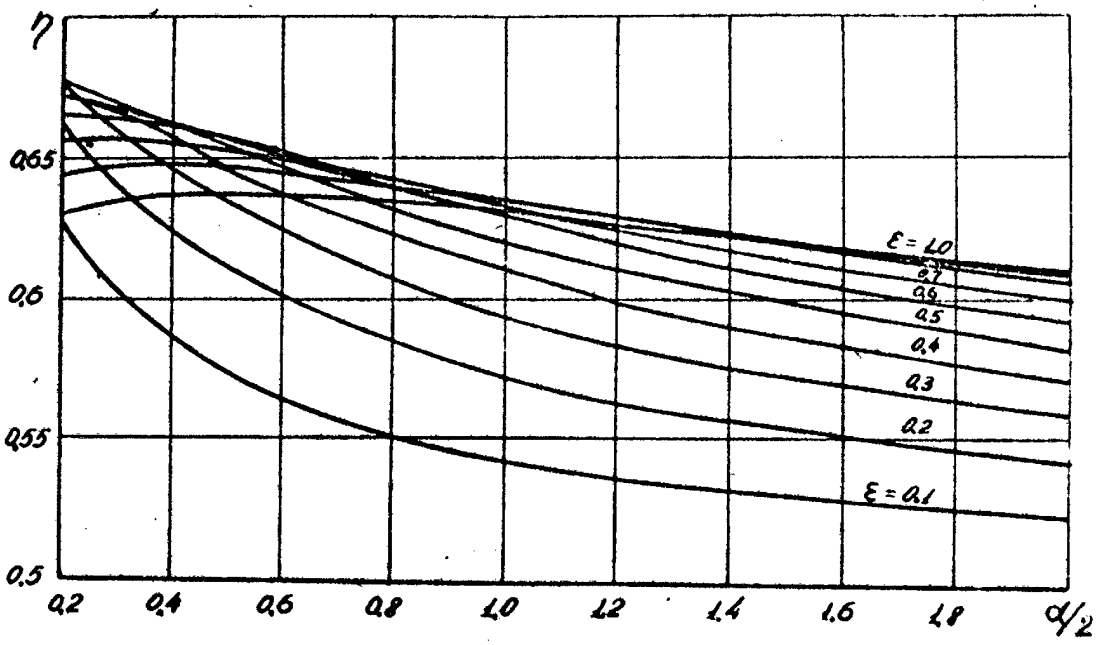


Fig. 15

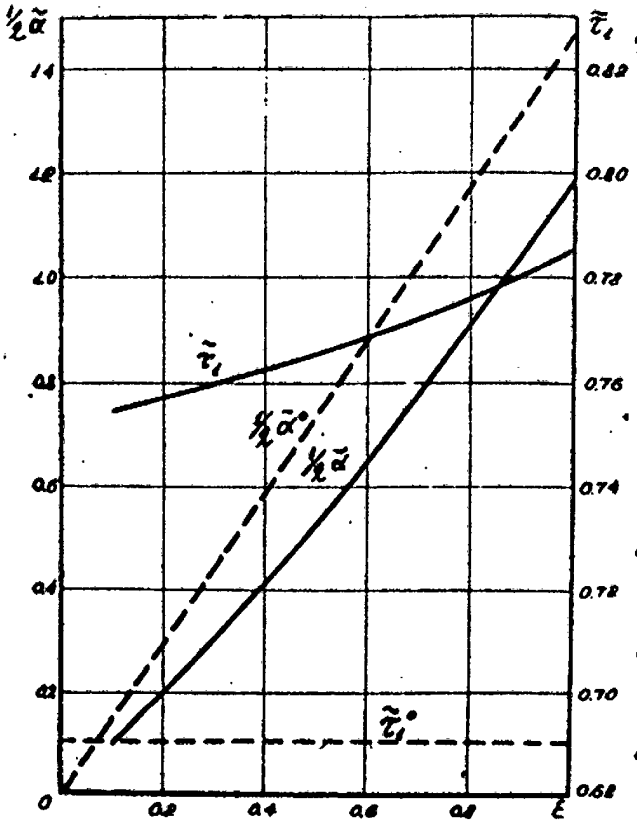


Fig.16

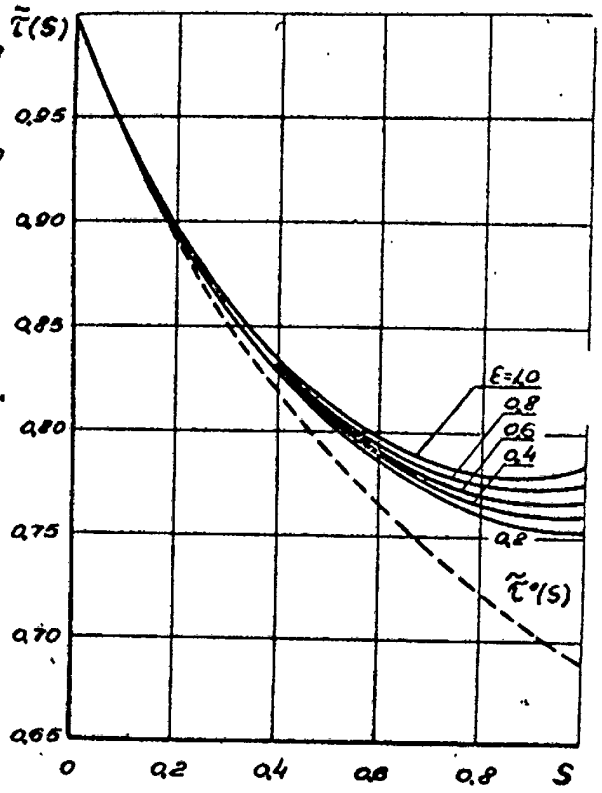


Fig.17

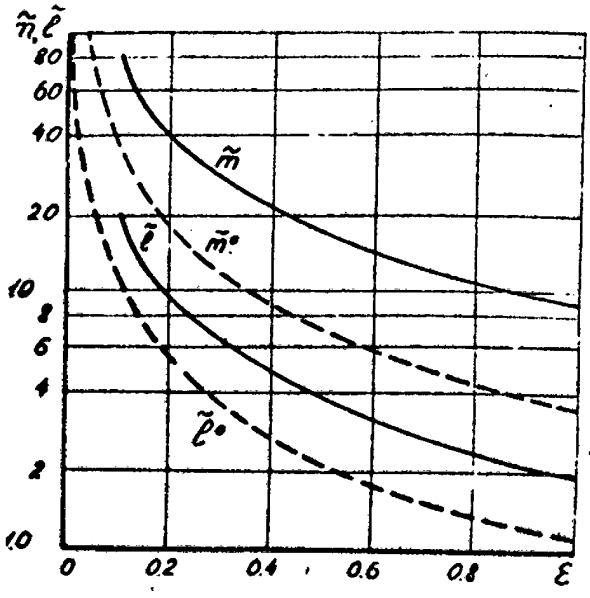


Fig.18

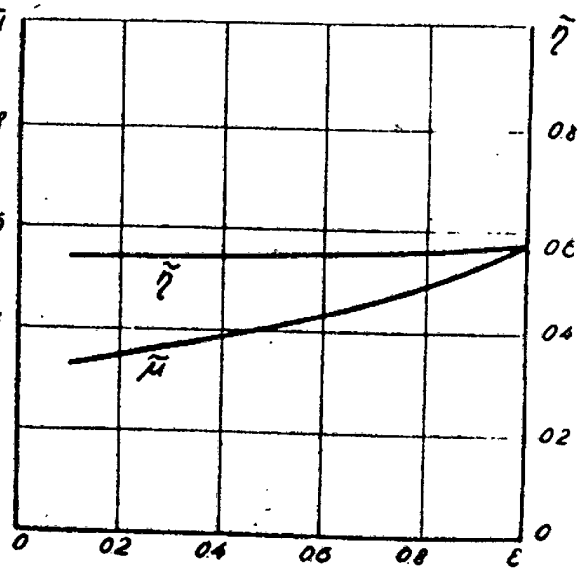


Fig.19

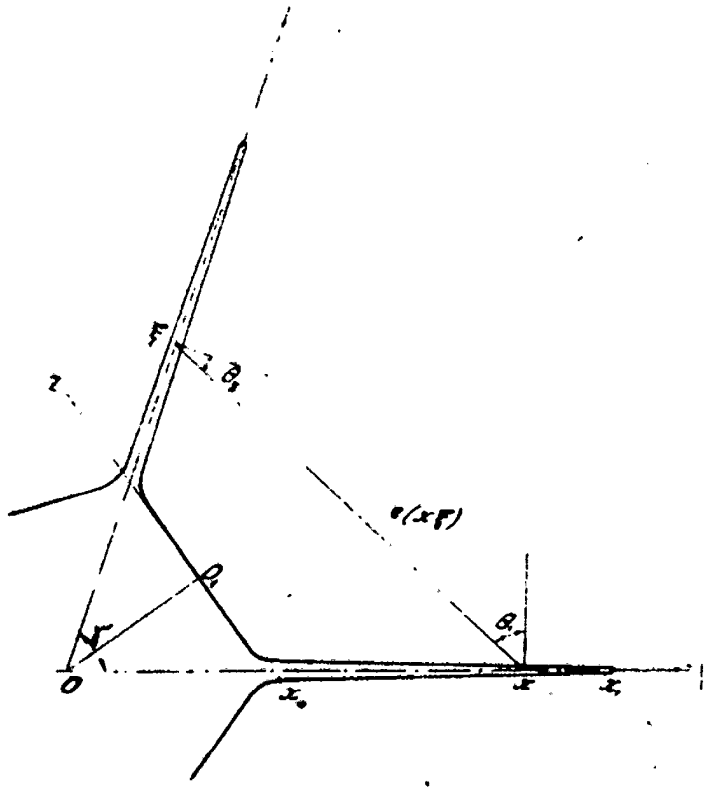


Fig. 20

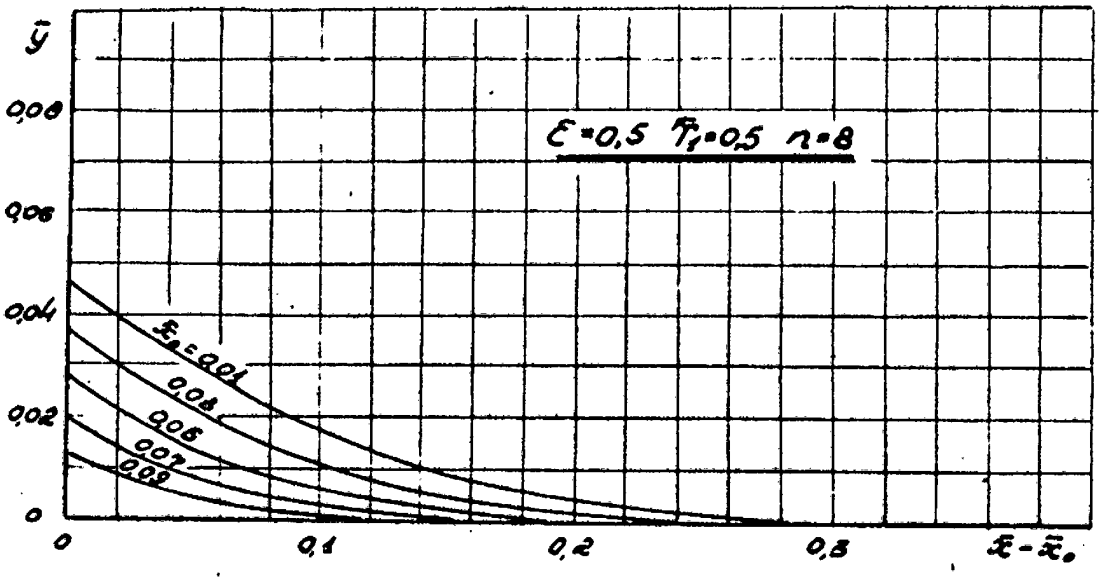


Fig. 21

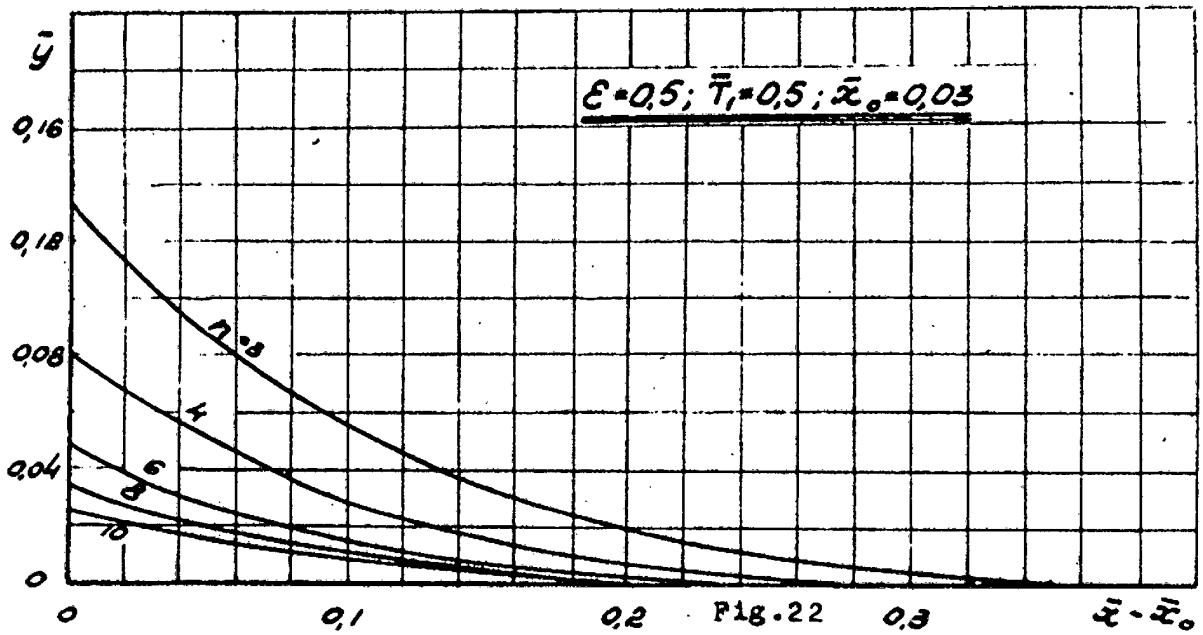


Fig.22

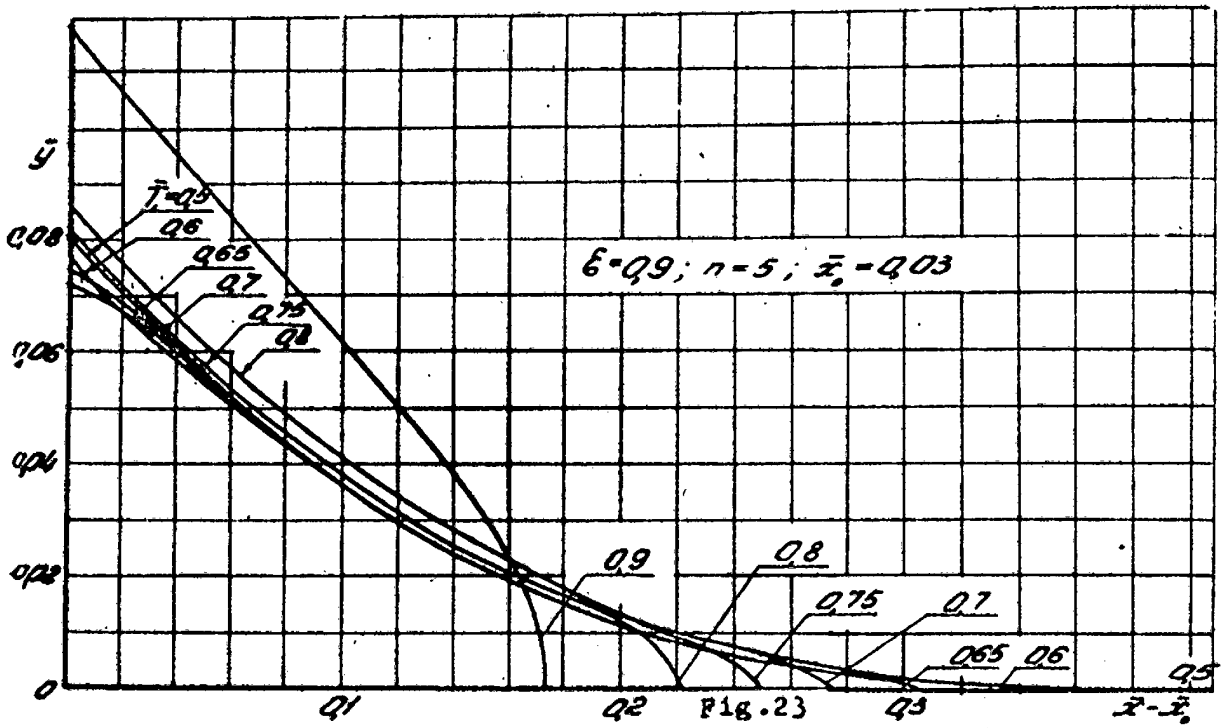


Fig.23

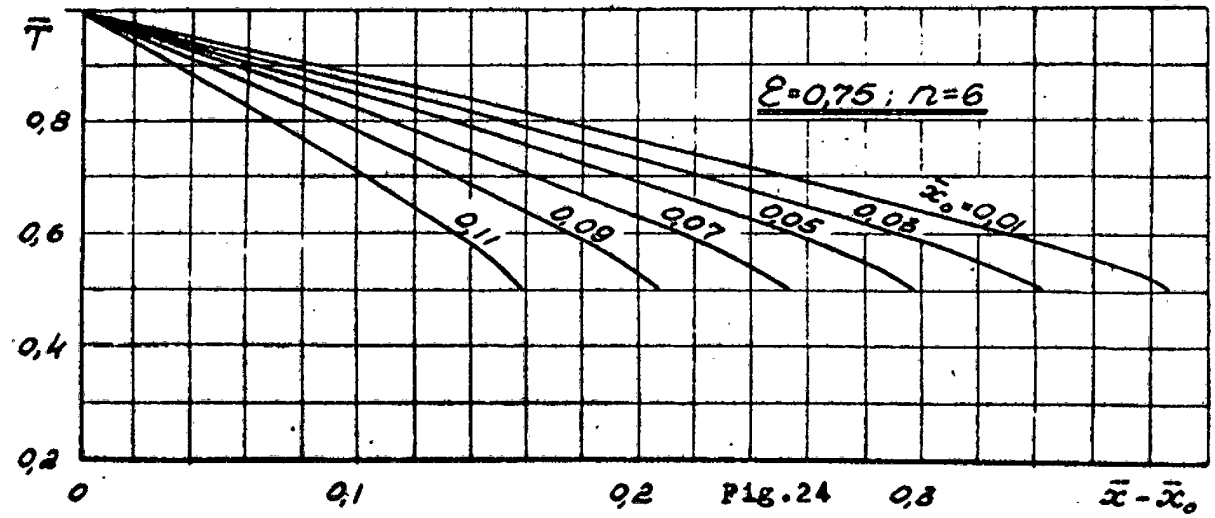


Fig.24

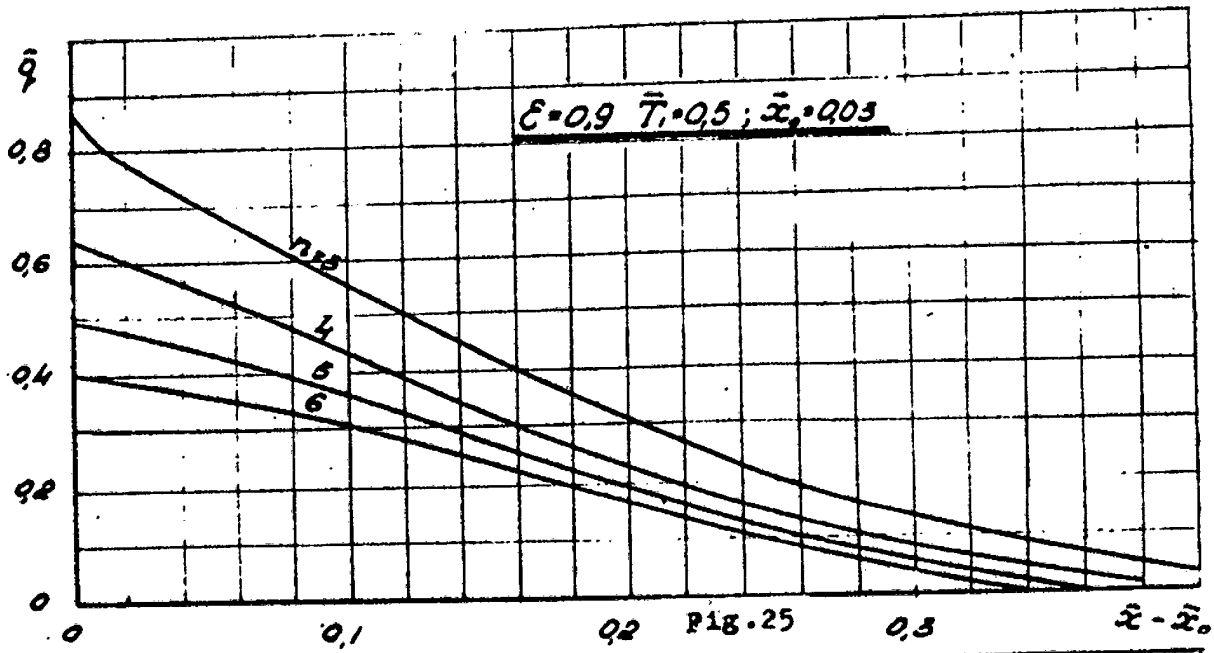


Fig. 25

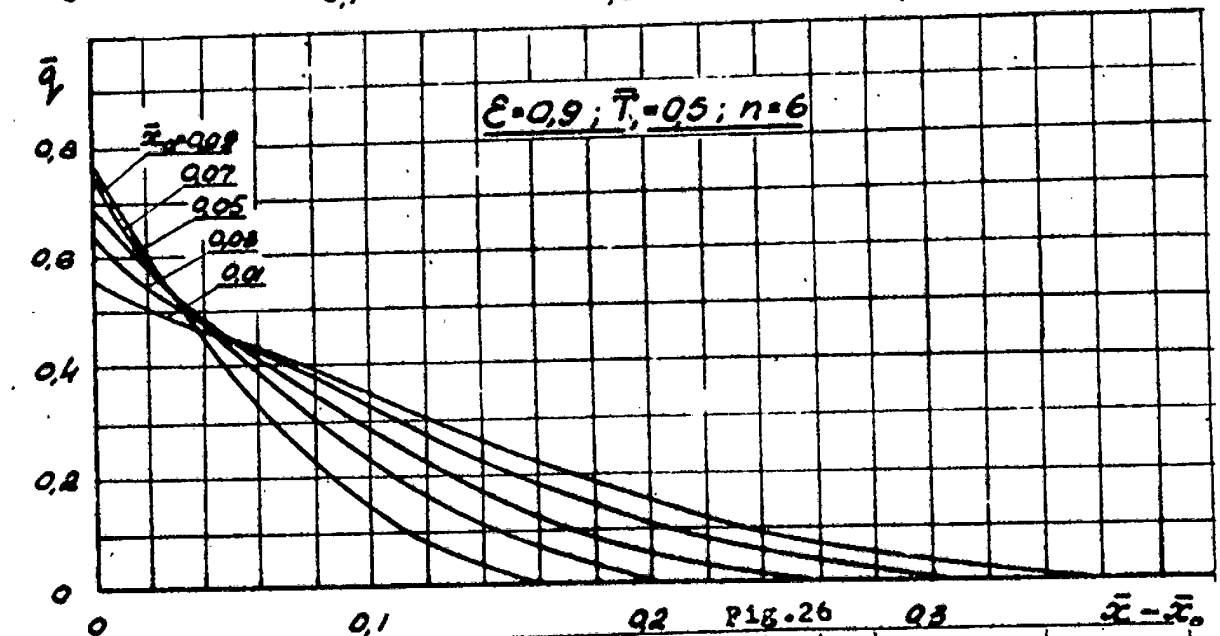


Fig. 26

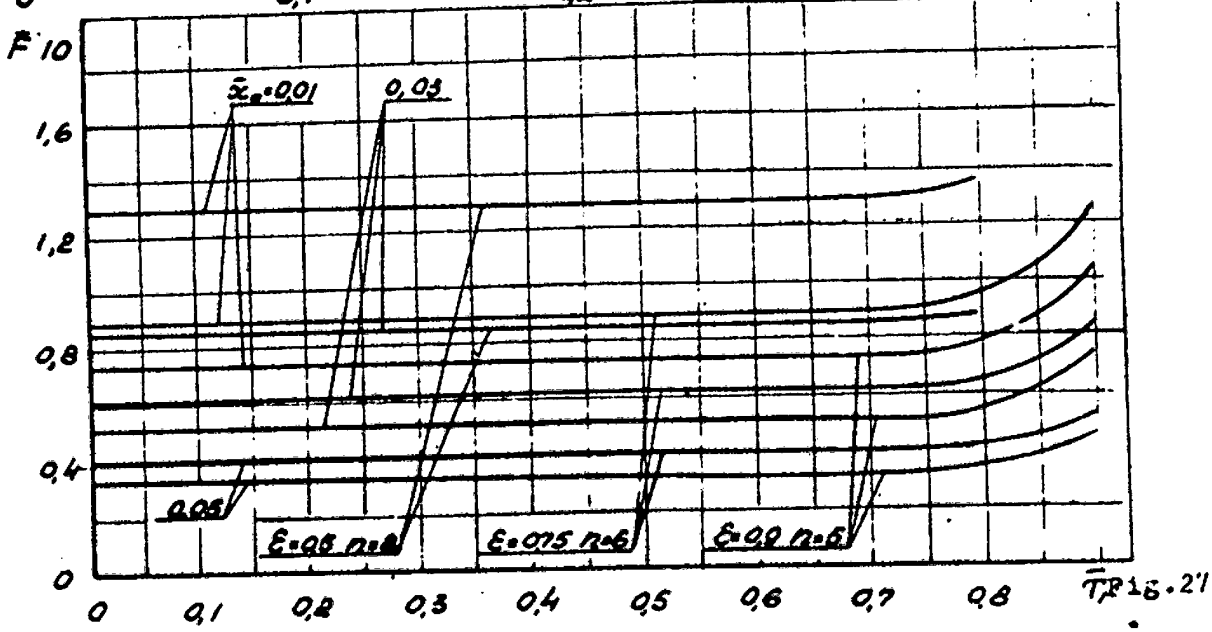


Fig. 27

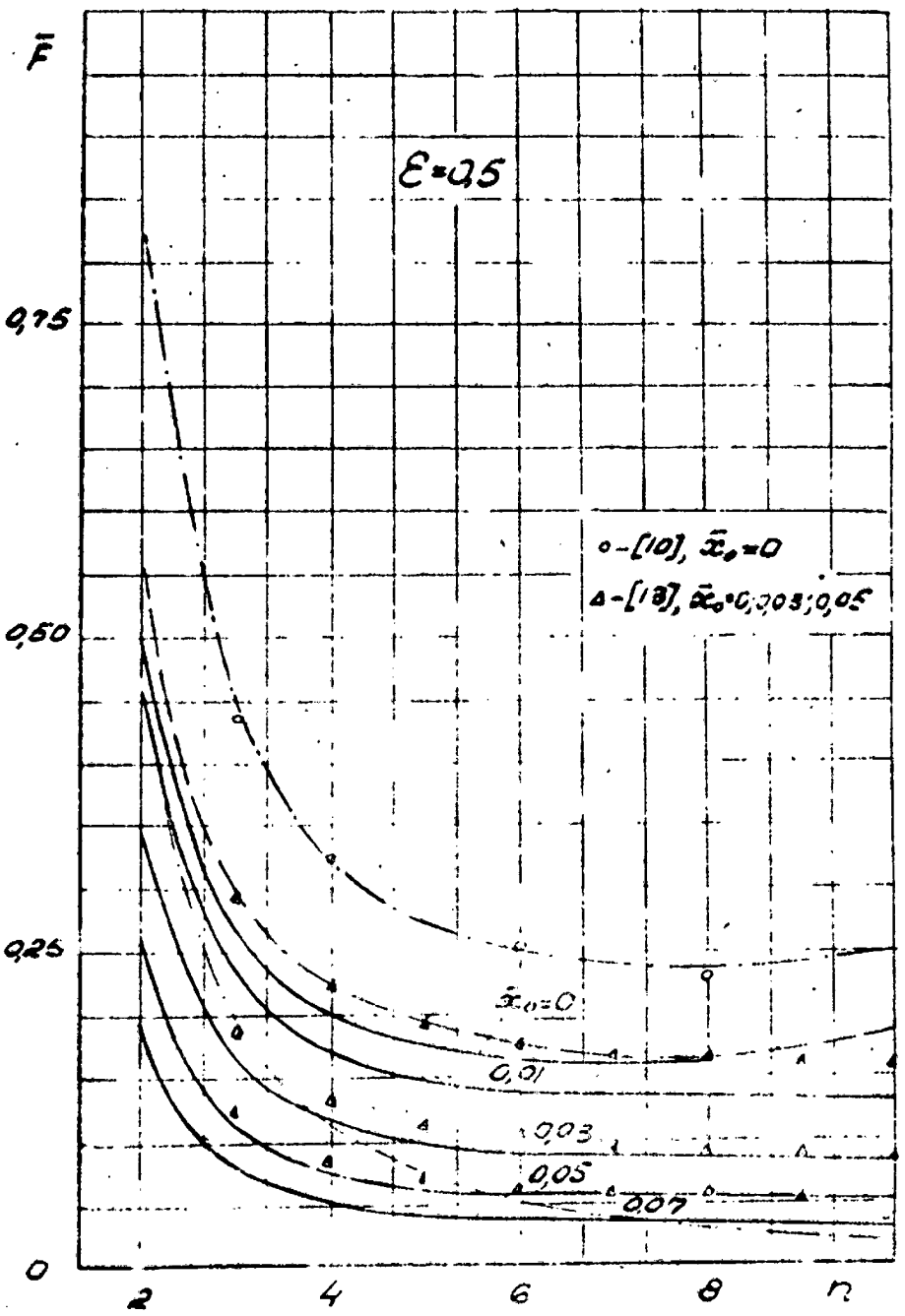


Fig. 28

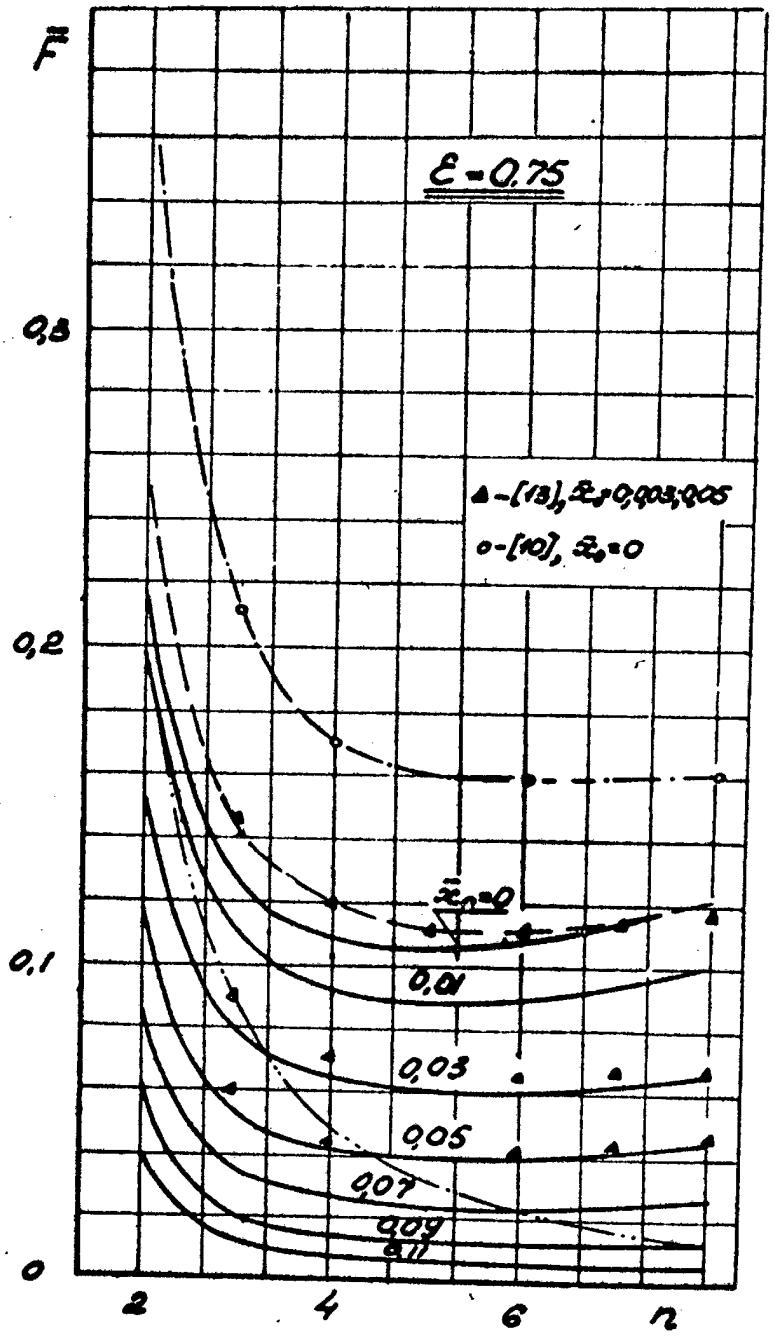


Fig. 29

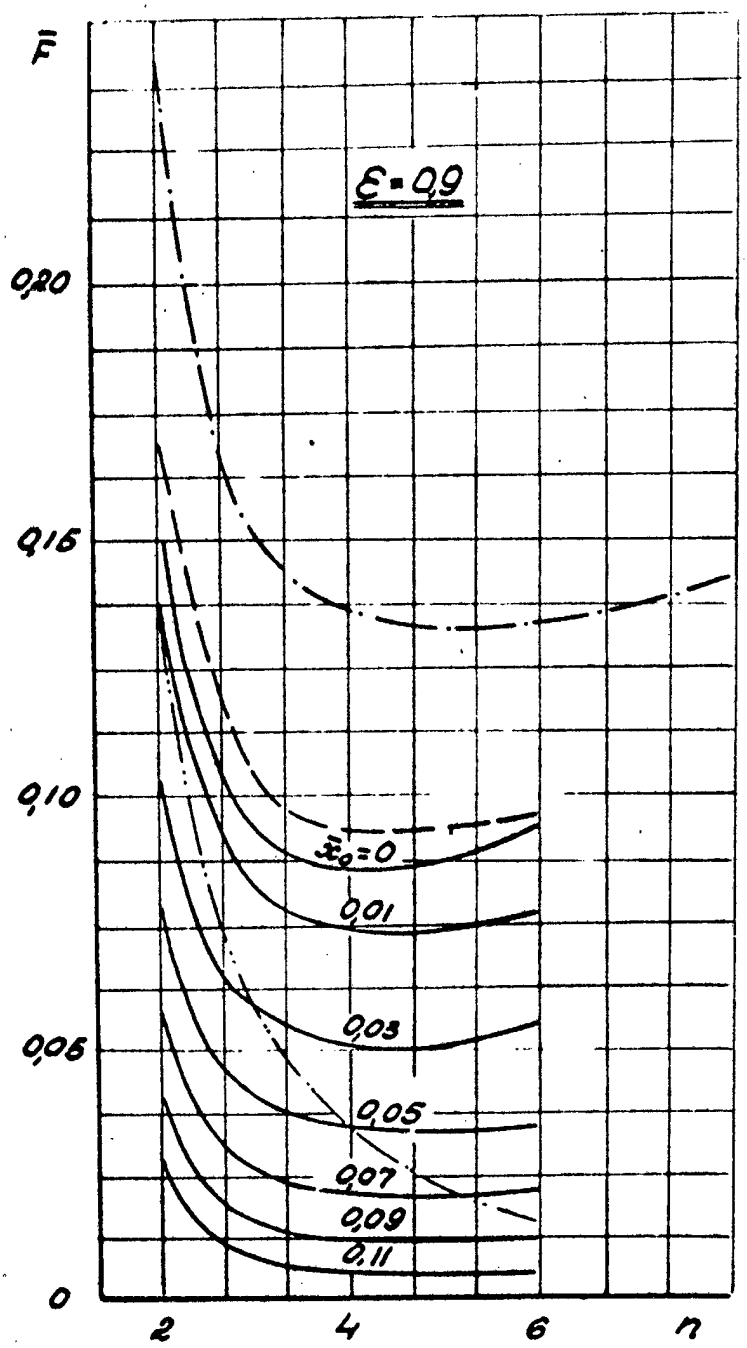


Fig.30

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