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CONTRIBUTION TO THE THEORY OF STRONG INTERACTION OF A BOUNDARY LAYER WITH AN INVISCID HYPERSONIC FLOW

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SUMMARY

The paper discusses the higher-order approximations in the theory of strong interaction between a boundary layer and an external inviscid flow. Known results concerning the problem of the unsteady motion of a gas past an infinite plate, and the problem of the steady flow past a semi-infinite plate are refined. The analysis leads to asymptotic expressions for the transverse displacements of a plate, or its shape, that correspond to the pressure-distribution law of the first approximation.

INTRODUCTION

The effect of the viscosity and heat conductivity of a gas upon the flow field near a body moving at hypersonic speed is known to lend itself to an approximate analysis on the basis of the theory of boundary-layer interaction with the external viscous region of the flow /1/. If the body is sufficiently slender, and the Mach number and Reynolds number of the problem are such that the ratio $M_\infty^3 / \sqrt{Re_\infty} \gg 1$, then a strong interaction takes place, in which the pressure field in the perturbed region of the flow is primarily influenced by the displacing effect of the boundary layer, and to a much lesser degree by the shape of the surface of the body that is situated in the flow. The most typical examples of a plane flow of this type, namely, the unsteady flow past an infinite plane that has been abruptly set in motion at constant speed, and the steady flow past a semi-infinite plate, have been discussed in /2, 3/.

The solutions obtained in these papers are based on combining the exact (self-similar) solutions of the boundary-layer equations with the solutions of the equations in the small-perturbation theory of hypersonic flow; the procedure of combining these solutions having been developed only in the first approximation. The result of this, is the peculiar behavior of the solutions in the intermediate region (at the external boundary of the boundary layer), which manifests itself in that the enthalpy of the gas in this region tends to zero, while the density undergoes an infinite increase.

The papers /2, 3/ include also accuracy estimates of the first-approximation theory.

The object of the present work is to develop the higher-order approximations to these problems, or more precisely, to problems associated with the asymptotic behavior of the flow field of a viscous heat-conducting gas behind shock waves the propagation of which is controlled by the same law ($y \sim t^{3/4}$ and $y \sim x^{3/4}$), in the limiting case where $M_\infty \to \infty$.

I. NONSTEADY MOTION

1. Let us examine the one-dimensional nonsteady motion of a viscous heat-conducting gas, caused by an infinite plate that has been set in motion at a velocity that has a constant longitudinal component U_∞. We assume a linear dependence of the viscosity coefficient of the gas upon the specific enthalpy:

$$\mu = C U_\infty^2 h \qquad \text{(I.1)}$$

In this case, the Navier-Stokes equations may be written in the form

$$\rho\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(h\frac{\partial u}{\partial y}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \frac{4}{3}\frac{\partial}{\partial y}\left(h\frac{\partial v}{\partial y}\right)$$

$$\rho\left(\frac{\partial h}{\partial t} + v\frac{\partial h}{\partial y}\right) = \frac{\partial p}{\partial t} + v\frac{\partial p}{\partial y} + \frac{1}{\sigma}\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right) + h\left(\frac{\partial u}{\partial y}\right)^2 + \frac{4}{3}h\left(\frac{\partial v}{\partial y}\right)^2 \qquad \text{(I.2)}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial y} = 0, \qquad p = \frac{\gamma-1}{\gamma}\rho h,$$

where the velocity-vector components u, v refer to the longitudinal velocity of the plate, the pressure p refers to the quantity $\rho_\infty U_\infty$, the density ρ refers to the density of the unperturbed flow ρ_∞, the specific enthalpy h refers to the quantity U_∞^2; the dimensionless independent variables t, y refer to the quantities $\frac{C}{\rho_\infty}$ and $\frac{C U_\infty}{\rho_\infty}$, respectively; σ and γ are respectively the Prandtl number and the ratio of specific heats of the gas.

Introducing, on the basis of the continuity equation, the function Ψ, defined by the relations

$$\frac{\partial \Psi}{\partial t} = -\rho v, \qquad \frac{\partial \Psi}{\partial y} = \rho, \qquad \text{(I.3)}$$

we write the system (1.2) in terms of the independent variables t, ψ. As a result we get

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial \psi}\left(\rho h \frac{\partial u}{\partial \psi}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial p}{\partial \psi} = \frac{4}{3}\frac{\partial}{\partial \psi}\left(\rho h \frac{\partial v}{\partial \psi}\right) \tag{I.4}$$

$$\rho\frac{\partial h}{\partial t} = \frac{\partial p}{\partial t} + \frac{1}{6}\rho\frac{\partial}{\partial \psi}\left(\rho h \frac{\partial h}{\partial \psi}\right) + \rho^2 h\left(\frac{\partial u}{\partial \psi}\right)^2 + \frac{4}{3}\rho^2 h\left(\frac{\partial v}{\partial \psi}\right)^2$$

$$\rho\frac{\partial y}{\partial \psi} = 1, \qquad \frac{\partial y}{\partial \psi} = v, \qquad p = \frac{\gamma-1}{\gamma}\rho h.$$

As already stated, our object is to obtain an asymptotic solution of these equations that corresponds to the one-dimensional motion of a gas propagating according to the law

$$y = ct^{\frac{3}{4}} \tag{I.5}$$

that satisfies the condition of attachment

$$u = 1 \tag{I.6}$$

and the condition of the absence of a heat flux

$$\frac{\partial h}{\partial \psi} = 0 \tag{I.7}$$

on the plate surface $\psi = 0$, which is thus postulated as a heat insulated surface.

2. For the external portion of the flow field, adjoining to the surface of the shock wave, such a solution is known to have the form

$$\begin{aligned} y &= t^{\frac{3}{4}} Y_0(\nu) \\ u &= 0 \\ v &= t^{-\frac{1}{4}} V_0(\nu) \\ p &= t^{-\frac{1}{2}} P_0(\nu) \\ \rho &= R_0(\nu) \\ h &= t^{-\frac{1}{2}} H_0(\nu), \end{aligned} \tag{2.I}$$

where the independent variable

$$\nu = \psi t^{-\frac{3}{4}} \qquad (2.2)$$

Substituting the expression (2.1) into the initial system of equations (1.4), and retaining the principal terms in these equations, we obtain a system of ordinary differential equations for the well-known progressive motion of an inviscid gas:

$$\begin{aligned} &\frac{3}{4}\nu V_o' + \frac{1}{4} V_o = P_o' \\ &R_o\left(\frac{3}{2}\nu H_o' + H_o\right) = \frac{3}{2}\nu P_o' + P_o \\ &R_o Y_o' = 1 \\ &\frac{3}{4}\nu Y_o' - \frac{3}{4} Y_o + V_o = 0 \\ &P_o = \frac{\gamma - 1}{\gamma} R_o H_o . \end{aligned} \qquad (2.3)$$

We should note that taking a gas in the external region of the flow to be inviscid and nonconducting involves a relative error on the order of t^{-1}, since this is the order of smallness of the relation of the terms neglected in equations (1.4) to the principal terms.

The solution of the system of equations (2.3) must satisfy the set of boundary conditions at the surface of the shock wave, the propagation of which follows the equation (1.5). In the limiting case of a flow at $M_\infty \to \infty$, these boundary conditions take the form:

$$\begin{aligned} &Y_o(c) = c \\ &V_o(c) = \frac{3c}{2(\gamma+1)} \\ &P_o(c) = \frac{9c^2}{8(\gamma+1)} \\ &R_o(c) = \frac{\gamma+1}{\gamma-1} \\ &H_o(c) = \frac{9\gamma c^2}{8(\gamma+1)^2}, \end{aligned} \qquad (2.4)$$

where the constant C has to be determined.

For the following, it will be essential to have the expressions for the sought functions of the external flow at $\nu \to 0$. To obtain these equations it is necessary to note that the second of the equations (2.3) is integrable with the aid of the last equation, giving:

$$\frac{P_0}{R_0^{\gamma}} = A_0 \nu^{-\frac{2}{3}} \tag{2.5}$$

where the constant of integration A_0 is determined from the boundary conditions (2.4):

$$A_0 = \frac{9C^{\frac{8}{3}}}{8(\gamma+1)} \left(\frac{\gamma-1}{\gamma+1}\right)^{\gamma}. \tag{2.6}$$

Making use of (2.5) and the remaining equations of the system (2.3), it is now easy to obtain the following expressions, valid for $\nu \to 0$:

$$\begin{aligned} Y_0 &= Y_{00} + Y_{01}\nu^{1-\frac{2}{3\gamma}} + O\left(\nu^{2-\frac{2}{3\gamma}}\right) \\ V_0 &= V_{00} + V_{01}\nu^{1-\frac{2}{3\gamma}} + O\left(\nu^{2-\frac{2}{3\gamma}}\right) \\ P_0 &= P_{00} + O(\nu) \\ R_0 &= R_{00}\nu^{\frac{2}{3\gamma}} + O\left(\nu^{1+\frac{2}{3\gamma}}\right) \\ H_0 &= H_{00}\nu^{-\frac{2}{3\gamma}} + O\left(\nu^{1-\frac{2}{3\gamma}}\right) \end{aligned} \tag{2.7}$$

The coefficients in these formulas are related by the following relations:

$$\begin{aligned} Y_{01} &= \frac{3\gamma}{3\gamma-2} A_0^{\frac{1}{\gamma}} P_{00}^{-\frac{1}{\gamma}}, \quad V_{00} = \frac{3}{4} Y_{00}, \\ V_{01} &= \frac{3}{2(3\gamma-2)} A_0^{\frac{1}{\gamma}} P_{00}^{-\frac{1}{\gamma}}, \quad R_{00} = A_0^{-\frac{1}{\gamma}} P_{00}^{\frac{1}{\gamma}}, \\ H_{00} &= \frac{\gamma}{\gamma-1} A_0^{\frac{1}{\gamma}} P_{00}^{1-\frac{1}{\gamma}} \end{aligned} \tag{2.8}$$

-3. To investigate the internal region of the flow field we, as usual, introduce the independent variable

$$\mathcal{N}=\Psi t^{-\frac{1}{4}}. \qquad (3.I)$$

To determine the form of the asymptotic expansion in this region, we express the external flow function through the independent variable of the external expansion

$$\nu = \mathcal{N} t^{-\frac{1}{2}} \qquad (3.2)$$

and consider the limit $t \to \infty$ at fixed values of $\mathcal{N}$.

Using the expressions (2.7), we get:

$$\begin{aligned}
y &= t^{\frac{3}{4}}\left[Y_{00}+Y_{01}\mathcal{N}^{1-\frac{2}{3\gamma}}t^{-\frac{1}{2}+\frac{1}{3\gamma}}+O\left(t^{-1+\frac{1}{3\gamma}}\right)\right]\\
u &= O\left(t^{-1}\right)\\
\vartheta &= t^{-\frac{1}{4}}\left[V_{00}+V_{01}\mathcal{N}^{1-\frac{2}{3\gamma}}t^{-\frac{1}{2}+\frac{1}{3\gamma}}+O\left(t^{-1+\frac{1}{3\gamma}}\right)\right]\\
p &= t^{-\frac{1}{2}}\left[P_{00}+O\left(t^{-\frac{1}{2}}\right)\right]\\
\rho &= R_{00}\mathcal{N}^{\frac{2}{3\gamma}}t^{-\frac{1}{3\gamma}}+O\left(t^{-\frac{1}{2}-\frac{1}{3\gamma}}\right)\\
h &= H_{00}\mathcal{N}^{-\frac{2}{3\gamma}}t^{-\frac{1}{2}+\frac{1}{3\gamma}}+O\left(t^{-1+\frac{1}{3\gamma}}\right)
\end{aligned} \qquad (3.3)$$

These expressions predict the form in which to seek the asymptotic solution for the internal region of the flow, namely:

$$\begin{aligned}
y &= t^{\frac{3}{4}}\left[y_0(\mathcal{N})+t^{-\frac{1}{2}+\frac{1}{3\gamma}}y_1(\mathcal{N})+\cdots\right]\\
u &= u_0(\mathcal{N})+t^{-\frac{1}{2}+\frac{1}{3\gamma}}u_1(\mathcal{N})+\cdots\\
\vartheta &= t^{-\frac{1}{4}}\left[\vartheta_0(\mathcal{N})+t^{-\frac{1}{2}+\frac{1}{3\gamma}}\vartheta(\mathcal{N})+\cdots\right]\\
p &= t^{-\frac{1}{2}}\left[p_0(\mathcal{N})+t^{-\frac{1}{2}+\frac{1}{3\gamma}}p_1(\mathcal{N})+\cdots\right]\\
\rho &= t^{-\frac{1}{2}}\left[\rho_0(\mathcal{N})+t^{-\frac{1}{2}+\frac{1}{3\gamma}}\rho_1(\mathcal{N})+\cdots\right]\\
h &= h_0(\mathcal{N})+t^{-\frac{1}{2}+\frac{1}{3\gamma}}h_1(\mathcal{N})+\cdots
\end{aligned} \qquad (3.4)$$

Indeed, the combination of the internal and external expansions can be now performed if, in conformity with the simplest form of the combination principle, the following boundary conditions for the external expansion functions are satisfied:

$$\left.\begin{aligned} y_0(N) &\to Y_{00} \\ u_0(N) &\to 0 \\ p_0(N) &\to P_{00} \\ h_0(N) &\to 0 \end{aligned}\right\} \quad \text{at} \quad N \to \infty \qquad (3.5)$$

in the first approximation, and

$$\left.\begin{aligned} y_1(N) &\to Y_{01}\, N^{1-\frac{2}{3\gamma}} \\ u_1(N) &\to 0 \\ p_1(N) &\to 0 \\ h_1(N) &\to H_{00}\, N^{-\frac{2}{3\gamma}} \end{aligned}\right\} \quad \text{at} \quad N \to \infty \qquad (3.6)$$

in the second approximation.

4. Substituting the expansion (3.4) into the initial equations (1.4), and retaining the principal terms, we obtain a system of equations for the first approximation, which may be written in the form:

$$\begin{gathered} p_0 = \frac{\gamma-1}{\gamma}\rho_0 h_0 = \text{const} \\ u_0'' + \frac{\gamma-1}{4\gamma p_0} N u_0' = 0 \\ \frac{\gamma}{\gamma-1}\frac{p_0}{6} h_0'' + \frac{1}{4} N h_0' - \frac{\gamma-1}{2\gamma} h_0 = -\frac{\gamma}{\gamma-1} p_0 u_0'^2 \\ y_0' = \frac{\gamma-1}{\gamma}\frac{h_0}{p_0} \\ v_0' = \frac{3}{4} y_0 - \frac{\gamma-1}{4\gamma} N \frac{h_0}{p_0}\,. \end{gathered} \qquad (4.\mathrm{I})$$

The boundary conditions for these equations are the conditions (3.5) as well as the conditions on the plate surface, which we will write in the form

$$u_0(0) = 1 \quad , \quad y_0(0) = h_0'(0) = 0 \tag{4.2}$$

i.e., our requirement, in addition to the fulfilment of the boundary conditions (1.6), (1.7), is that the plate be capable of displacements in its own plane, in the first approximation. The first-approximation problem, thus formulated, completely coincides with that discussed in /2/. Its solution is relatively simply. First, we note that the second of the ~~the~~ equations (4.1) is integrable in quadratures. Its particular solution that satisfies the boundary conditions is

$$u_0 = 1 - \sqrt{\frac{\gamma-1}{2\pi\gamma p_0}} \int_0^{N} e^{-\frac{\gamma-1}{8\gamma p_0}N^2} dN . \tag{4.3}$$

Having this, the third equation can be integrated. However, this is not required for the determination of the pressure distribution over the surface of the plate. To solve this problem, it is enough to find the expression for $y_0(N)$ at $N \to \infty$. On the basis of the fourth of the equations (4.1), we have:

$$\lim_{N\to\infty} y_0(N) = \frac{\gamma-1}{\gamma p_0} \int_0^{\infty} h_0 \, dN . \tag{4.4}$$

The integral in this expression is easy to calculate with the aid of the third equation of the system (4.1), if the boundary conditions for $h_0(N)$ and the exponential way ~~with~~ in which this function tends to zero at $N \to \infty$ are taken into account (see /2/). The result is the following expression:

$$\lim_{N\to\infty} y_0(N) = \frac{2\gamma}{3\gamma-2} \sqrt{\frac{\gamma-1}{\pi\gamma p_0}} \tag{4.5}$$

Making use of the boundary conditions (3.5), the expression is written in the form:

$$Y_{00} = \frac{2\gamma}{3\gamma-2} \sqrt{\frac{\gamma-1}{\pi\gamma P_{00}}} \tag{4.6}$$

The relation obtained constitutes the missing boundary condition for the system of equations for the inviscid external flow. The fulfilment of this condition uniquely determines the constant C in the shock-wave equation (1.5) and in the boundary conditions (2.4), thereby completely closing the first-approximation problem.

5. Let us examine the problem of the second approximation. Substituting the expansions (3.4) into the system (1.4), and equating the corresponding terms of the expansion, we get a system of linear differential equations for the second-approximation functions.

The second and the last of the equations (1.4), together with the boundary conditions (3.6) yield

$$p_1 = \frac{\gamma-1}{\gamma}\left(\rho_0 h_1 + h_0 \rho_1\right) = 0 \qquad (5.1)$$

Subsequently, on the basis of the first of the equations (1.4) and the boundary conditions (3.6) for u_1 we find that

$$u_1 = 0 \qquad (5.2)$$

Then, the equation for determining the function h_1, after simple transformations, taking (5.1), (5.2) and the results of the first approximation into account, takes the form:

$$\frac{\gamma}{\gamma-1}\frac{p_0}{\sigma} h_1'' + \frac{1}{4} N h_1' + \frac{1}{6\gamma} h_1 = 0 \qquad (5.3)$$

Its solution should satisfy the last of the boundary conditions (3.6) as well as the condition (1.7) on the heat-insulated surface:

$$h_1(N) \to H_{00} N^{-\frac{2}{3\gamma}} \quad \text{at} \quad N \to \infty \qquad (5.4)$$

$$h_1'(0) = 0$$

Finally, the equations for the function y_1 have the form:

$$y_1' = \frac{\gamma-1}{\gamma}\frac{h_1}{p_0}; \qquad (5.5)$$

Here, the function $y_1(N)$ must satisfy the first of the conditions (3.6)[x)].

By successive integration of (5.3) and (5.5) we find the value of function $y_1(0)$, that determines the transverse displacement of the plate:

$$y \cong t^{\frac{1}{4}+\frac{1}{3\gamma}} \cdot y_1(0) \tag{5.6}$$

Note, that in the derivation of the equations of the second approximation, in the initial equations were neglected the terms, the relation of which to the retained terms was of the order of $t^{-\frac{1}{2}+\frac{1}{3\gamma}}$, while in obtaining the expansions (3.4) on the basis of (3.3), the order of the highest of the neglected terms was $t^{-\frac{1}{2}}$. Hence, the solution of the problem under study, in the second approximation holds at a relative error of the order of $t^{-1+\frac{2}{3\gamma}}$ or $t^{-\frac{1}{2}}$, while the relative error of the first approximation is of the order of $t^{-\frac{1}{2}+\frac{1}{3\gamma}}$.

6. Summarizing the results obtained, we write the final equations for the pressure $\bar{p}=\rho_\infty U_\infty^2 p$ at the surface of the plate, and the rate of the transverse displacement of the plate $\bar{v}$. For this purpose, we will examine the Reynolds number of the problem, defined as

$$Re_\infty = \frac{\rho_\infty U_\infty^2 \bar{t}}{\mu_\infty}, \tag{6.I}$$

where $\bar{t}$ is dimensional time. As a result we find that

$$\frac{\bar{p}}{p_\infty} \cong \gamma\sqrt{\gamma-1}\, P_{00} \frac{M_\infty^3}{\sqrt{Re_\infty}}, \tag{6.2}$$

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x) It is readily shown that the asymptotic nature of the behavior of the functions of the internal expansion at $N \to \infty$, prescribed by the boundary conditions (3.5'), (3.6), completely corresponds to that deduced from a direct analysis of the differential equations for these functions.

and that the quantity

$$\frac{\bar{\vartheta}}{U_\infty} \cong \left(\frac{1}{4} + \frac{1}{3\gamma}\right) y_1(0) \left[\frac{\sqrt{\gamma - 1}\, M_\infty}{\sqrt{Re_\infty}}\right]^{\frac{3}{2} - \frac{2}{3\gamma}} \qquad (6.3)$$

Thus, the formula (6.3) defines the rate of the transverse displacement of the plate, the surface pressure of which varies according to the law (6.2).

II. STEADY-STATE MOTION

7. The equations of the plane steady motion of a viscous heat-conducting gas can be written in the following dimensionless form:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\frac{\partial p}{\partial x}=\frac{\partial}{\partial x}\left[h\left(\frac{4}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[h\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right],$$
$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+\frac{\partial p}{\partial y}=\frac{\partial}{\partial y}\left[h\left(\frac{4}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial u}{\partial x}\right)\right]+\frac{\partial}{\partial x}\left[h\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right],$$
$$\rho\left(u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}\right)=u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+\frac{1}{\sigma}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right)+\frac{1}{\sigma}\frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)+ \qquad (7.1)$$
$$+2h\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+h\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2-\frac{2}{3}h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2,$$
$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0, \qquad p=\frac{\gamma-1}{\gamma}\rho h.$$

Here, the velocity-vector components refer to the velocity of the unperturbed flow U_∞, the pressure refers to the double velocity head $\rho_\infty U_\infty^2$, the density refers to the density of the unperturbed flow ρ_∞, the specific enthalpy refers to the quantity U_∞^2. The independent variables refer to the characteristic length

$$L=\frac{C\,U_\infty}{\rho_\infty} \qquad (7.2)$$

where C is the proportionality factor in the relation of the viscosity coefficient to the enthalpy, which as before we will consider as the linear relation (1.1). Introducing the stream function ψ, defined by the relations

$$\frac{\partial \psi}{\partial x}=-\rho v, \qquad \frac{\partial \psi}{\partial y}=\rho u \qquad (7.3)$$

we write equation (7.1) in terms of the independent variables x, ψ, getting a system of the form:

$$\rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} - \rho v \frac{\partial p}{\partial \psi} = \left(\frac{\partial}{\partial x} - \rho v \frac{\partial}{\partial \psi}\right)\left[\frac{4}{3} h \left(\frac{\partial u}{\partial x} - \rho v \frac{\partial u}{\partial \psi}\right) - \right.$$
$$\left. - \frac{2}{3} \rho h u \frac{\partial v}{\partial \psi}\right] + \rho u \frac{\partial}{\partial \psi}\left[\rho h u \frac{\partial u}{\partial \psi} + h\left(\frac{\partial v}{\partial x} - \rho v \frac{\partial v}{\partial \psi}\right)\right] ,$$

$$\rho u \frac{\partial v}{\partial x} + \rho u \frac{\partial p}{\partial \psi} = \rho u \frac{\partial}{\partial \psi}\left[\frac{4}{3} \rho h u \frac{\partial v}{\partial \psi} - \frac{2}{3} h \left(\frac{\partial u}{\partial x} - \rho v \frac{\partial u}{\partial \psi}\right)\right] +$$
$$+ \left(\frac{\partial}{\partial x} - \rho v \frac{\partial}{\partial \psi}\right)\left[\rho h u \frac{\partial u}{\partial \psi} + h\left(\frac{\partial v}{\partial x} - \rho v \frac{\partial v}{\partial \psi}\right)\right] ,$$

$$\rho u \frac{\partial h}{\partial x} - u \frac{\partial p}{\partial x} = \frac{1}{\sigma}\left(\frac{\partial}{\partial x} - \rho v \frac{\partial}{\partial \psi}\right)\left[h\left(\frac{\partial h}{\partial x} - \rho v \frac{\partial h}{\partial \psi}\right)\right] + \tag{7.4}$$
$$+ \frac{1}{\sigma} \rho u \frac{\partial}{\partial \psi}\left(\rho h u \frac{\partial h}{\partial \psi}\right) + 2h\left[\left(\frac{\partial u}{\partial x} - \rho v \frac{\partial u}{\partial \psi}\right)^2 + \left(\rho u \frac{\partial v}{\partial \psi}\right)^2\right] +$$
$$+ h\left(\rho u \frac{\partial u}{\partial \psi} + \frac{\partial v}{\partial x} - \rho v \frac{\partial v}{\partial \psi}\right)^2 - \frac{2}{3} h \left(\frac{\partial u}{\partial x} - \rho v \frac{\partial u}{\partial \psi} + \rho u \frac{\partial v}{\partial \psi}\right)^2$$

$$\rho u \frac{\partial y}{\partial \psi} = 1 , \quad u \frac{\partial y}{\partial x} = v , \quad p = \frac{\gamma - 1}{\gamma} \rho h$$

Our problem consists in obtaining an asymptotic solution of these equations that corresponds to the steady motion of a gas behind a shock wave having the form:

$$y = c x^{\frac{3}{4}} \tag{7.5}$$

and which satisfies the boundary conditions on a heat-insulated semi-infinite surface $(\psi = 0)$, the form of which

$$y = f(x) \tag{7.6}$$

is to be determined. These conditions have the form:

$$u = v = 0 , \quad \frac{\partial h}{\partial \psi} = \frac{f'(x) \frac{\partial h}{\partial x}}{\rho\left[1 + f'^2(x)\right]} \tag{7.7}$$

8. Let us start by examining the asymptotic expansion that holds for the external part of the flow, in the limits of an approximation in which this region of the flow may be treated as inviscid. The expansions for this region are written:

$$y = \xi^{\frac{3}{4}}\left[Y_0(\nu) + \xi^{-\frac{1}{2}} Y_1(\nu) + \cdots\right]$$
$$u - 1 = \xi^{-\frac{1}{2}}\left[U_0(\nu) + \xi^{-\frac{1}{2}} U_1(\nu) + \cdots\right]$$
$$v = \xi^{-\frac{1}{4}}\left[V_0(\nu) + \xi^{-\frac{1}{2}} V_1(\nu) + \cdots\right] \qquad (8.I)$$
$$p = \xi^{-\frac{1}{2}}\left[P_0(\nu) + \xi^{-\frac{1}{2}} P_1(\nu) + \cdots\right]$$
$$\rho = R_0(\nu) + \xi^{-\frac{1}{2}} R_1(\nu) + \cdots$$
$$h = \xi^{-\frac{1}{2}}\left[H_0(\nu) + \xi^{-\frac{1}{2}} H_1(\nu) + \cdots\right]$$

where, the independent variables ξ, ν are defined by the relations

$$x = \xi$$
$$\psi = \xi^{\frac{3}{4}} n = \xi^{\frac{3}{4}}\left[\nu + \xi^{-\frac{1}{2}} \psi_1(\nu) + \cdots\right] \qquad (8.2)$$

We make use of the expansion of one of the independent variables to obtain by the method in /4/, a solution for the external inviscid flow, that is valid over the entire flow field, including the proximity of the plate surface. This is necessary because, in distinction from the first part of this paper, the first terms of the expansion (8.1) does not represent an exact solution of the problem for the external inviscid flow but merely its approximate solution which, for $\nu \to 0$, exhibits some peculiar features that the exact solution lacks.

On the basis of (8.2), we obtain the following formulas for the transformation of the derivatives to the independent variables ξ, ν:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{3}{4}\xi^{-1}\nu\frac{\partial}{\partial \nu} + \frac{3}{4}\xi^{-\frac{3}{2}}\left(\nu\Psi_1' - \frac{1}{3}\Psi_1\right)\frac{\partial}{\partial \nu} + \cdots$$
$$\frac{\partial}{\partial \Psi} = \xi^{-\frac{3}{4}}\frac{\partial}{\partial \nu} - \xi^{-\frac{5}{4}}\Psi_1'\frac{\partial}{\partial \nu} - \cdots \qquad (8.3)$$

The boundary conditions for the solution of the external problem are the conditions on the shock wave (7.5) which in the limiting case $M_\infty \to \infty$ have the form:

$$n = c \quad , \quad y = c\,\xi^{\frac{3}{4}}$$
$$u - 1 = -\frac{9c^2}{8(\gamma+1)}\xi^{-\frac{1}{2}}\left[1 - \frac{9}{16}c^2\xi^{-\frac{1}{2}} + 0\left(\xi^{-1}\right)\right] \qquad (8.4)$$
$$v = \frac{3c}{2(\gamma+1)}\xi^{-\frac{1}{4}}\left[1 - \frac{9}{16}c^2\xi^{-\frac{1}{2}} + 0\left(\xi^{-1}\right)\right]$$
$$p = \frac{9c^2}{8(\gamma+1)}\xi^{-\frac{1}{2}}\left[1 - \frac{9}{16}c^2\xi^{-\frac{1}{2}} + 0\left(\xi^{-1}\right)\right]$$
$$\rho = \frac{\gamma+1}{\gamma-1}$$
$$h = \frac{9\gamma c^2}{8(\gamma+1)^2}\xi^{-\frac{1}{2}}\left[1 - \frac{9}{16}c^2\xi^{-\frac{1}{2}} + 0\left(\xi^{-1}\right)\right]$$

Substituting the extensions (8.1), (8.3) into the initial system of equations (7.4) and the boundary conditions (8.4), and retaining the principal terms, we obtain the systems of differential equations and the boundary conditions of the first approximation, that are fully equivalent to the problem of the one-dimensional nonsteady motion of an inviscid gas (2.1), (2.4), ~~that has been~~ discussed in the first part of this paper. Hence, changing the designation of the independent variable t to ξ, we can make use of the corresponding formulas in § 2, without introducing any changes.

Then, for the longitudinal component of the velocity factor, we get in the first approximation:

$$U_0 + \frac{V_0^2}{2} + \frac{\gamma}{\gamma-1}\,\frac{P_0}{R_0} = 0 \qquad (8.5)$$

from which, for $\nu \to 0$, it follows that

$$U_0 = U_{00}\,\nu^{-\frac{2}{3\gamma}} + O(\nu^0), \tag{8.6}$$

where

$$U_{00} = -\frac{\gamma}{\gamma-1} A_0^{\frac{1}{\gamma}} P_{00}^{1-\frac{1}{\gamma}}. \tag{8.7}$$

9. After simple transformations using the relations of the first approximation, the second-approximation equations may be written in the form:

$$\begin{aligned}
&U_1 + V_0 V_1 + H_1 + \frac{U_0^2}{2} = 0\\
&\frac{3}{4}(\nu V_1)' - \frac{3}{4}\left(\nu \Psi_1' - \frac{1}{3}\Psi_1\right) V_0' = P_1' - \Psi_1 P_0'\\
&\frac{3}{2}\nu^2\left(\frac{P_1}{P_0} - \gamma\frac{\mathcal{R}_1}{\mathcal{R}_0}\right)' + \nu\left(\frac{P_1}{P_0} - \gamma\frac{\mathcal{R}_1}{\mathcal{R}_0}\right) = -\left(\nu\Psi_1' - \frac{1}{3}\Psi_1\right)\\
&Y_1' + \frac{1}{\mathcal{R}_0}\left(U_0 + \frac{\mathcal{R}_1}{\mathcal{R}_0}\right) = \Psi_1' Y_0'\\
&\frac{3}{4}\nu Y_1' - \frac{1}{4} Y_1 + V_1 - V_0 U_0 = \frac{3}{4}\left(\nu\Psi_1' - \frac{1}{3}\Psi_1\right) Y_0'\\
&P_1 = \frac{\gamma-1}{\gamma}\left(\mathcal{R}_0 H_1 + H_0 \mathcal{R}_1\right)
\end{aligned} \tag{9.I}$$

In order to eliminate from the second approximation the peculiar entropy features (at $\nu \to 0$) of a higher order than in the first approximation, following the method in /4/, in the fourth of the equations (9.1) we may set:

$$\Psi_1' = U_0 + \frac{\mathcal{R}_1}{\mathcal{R}_0} \tag{9.2}$$

It then takes the form:

$$Y_1' = 0 \tag{9.3}$$

Now, these two equations, together with the remaining equations (9.1), form a closed system.

On the basis of (8.4), the boundary conditions for this system of equations may be written in the form:

$$\Psi_1(c)=0 \quad , \quad Y_1(c)=0$$
$$U_1(c)=\frac{81c^4}{128(\gamma+1)}$$
$$V_1(c)=-\frac{27c^3}{32(\gamma+1)}$$
$$P_1(c)=-\frac{81c^4}{128(\gamma+1)} \tag{9.4}$$
$$R_1(c)=0$$
$$H_1(c)=-\frac{81\gamma c^4}{128(\gamma+1)^2}$$

By taking $\Psi_1(c)=0$, we have eliminated in the derivation of these boundary conditions the displacement of the lines of flow in the proximity of the shock wave.

The equation (9.3) together with the boundary conditions (9.4) yields:

$$Y_1(\nu)=0 \tag{9.5}$$

The second of the equations (9.1) can be now integrated. Its solution that satisfies the boundary conditions (9.4) has the form:

$$\frac{P_1}{P_0}-\gamma\frac{R_1}{R_0}=-\frac{9}{16}c^{\frac{8}{3}}\nu^{-\frac{2}{3}}-\frac{2}{3}\nu^{-1}\Psi_1(\nu) \tag{9.6}$$

Treating the equations (9.1), (9.2), (9.5), (9.6) simultaneously, and taking the results (2.5), (2.7) of the first approximation into account, it is possible to determine the behavior of the functions of the second approximation, at $\nu \to 0$. There approximate expressions for this region have the form:

$$\Psi_1 = \Psi_{10}\,\nu^{\frac{1}{3}} + o\left(\nu^{1-\frac{2}{3\gamma}}\right),$$
$$U_1 = U_{10}\,\nu^{-\frac{2}{3}-\frac{2}{3\gamma}} + o\left(\nu^{-\frac{4}{3\gamma}}\right),$$
$$V_1 = V_{10}\,\nu^{-\frac{2}{3\gamma}} + o\left(\nu^{1-\frac{4}{3\gamma}}\right),$$
$$P_1 = o\left(\nu^{\frac{1}{3}}\right) \qquad (9.7)$$
$$\mathcal{R}_1 = \mathcal{R}_{10}\,\nu^{-\frac{2}{3}+\frac{2}{3}\gamma} + o\left(\nu^{0}\right),$$
$$H_1 = H_{10}\,\nu^{-\frac{2}{3}-\frac{2}{3\gamma}} + o\left(\nu^{-\frac{4}{3\gamma}}\right),$$

The coefficients in these formulas are related to the coefficients of the functions of the first approximation (2.8) by the relations:

$$\Psi_{10} = -\frac{27C^{\frac{8}{3}}}{16(2-\gamma)}$$
$$U_{10} = -\frac{9\gamma C^{\frac{8}{3}}}{16(\gamma-1)(2-\gamma)}\,A_0^{\frac{1}{\gamma}}\,P_{00}^{1-\frac{1}{\gamma}}$$
$$V_{10} = -\frac{3\gamma}{4(\gamma-1)}\,A_0^{\frac{1}{\gamma}}\,Y_{00}\,P_{00}^{1-\frac{1}{\gamma}} \qquad (9.8)$$
$$\mathcal{R}_{10} = -\frac{9C^{\frac{8}{3}}}{16(2-\gamma)}\,A_0^{-\frac{1}{\gamma}}\,P_{00}^{\frac{1}{\gamma}}$$
$$H_{10} = \frac{9\gamma C^{\frac{8}{3}}}{16(\gamma-1)(2-\gamma)}\,A_0^{\frac{1}{\gamma}}\,P_{00}^{1-\frac{1}{\gamma}},$$

in which the constant A_0 is determined from formula (2.6).

10. In the internal region of the flow, the dimensionless independent variable of the order of unity, is

$$N = \Psi\,\xi^{-\frac{1}{4}} \qquad (10.1)$$

In order to determine the form of the solution in this region, let us express the functions of the internal flow field through the independent variable of the internal expansion

$$n = \mathcal{N}\xi^{-\frac{1}{2}}, \qquad (10.2)$$

and examine their behavior at $\xi \to \infty$ and a fixed value of $\mathcal{N}$. To this end, we first substitute the expansion for the indepent variable n (8.2) into (10.2), thereby obtaining the following relation between the independent variables of the internal and external expansions:

$$\nu = \mathcal{N}\xi^{-\frac{1}{2}} - \Psi_{10}\mathcal{N}^{\frac{1}{3}}\xi^{-\frac{2}{3}} + O\left(\xi^{-1+\frac{1}{3\gamma}}\right) \qquad (10.3)$$

Now, making use of the expressions (2.7), (8.6) and (9.7) for the functions of the first and second approximations contained in the expansion (8.1), we get

$$y = \xi^{\frac{3}{4}}\left[Y_{00} + Y_{01}\mathcal{N}^{1-\frac{2}{3\gamma}}\xi^{-\frac{1}{2}+\frac{1}{3\gamma}} - \left(1-\frac{2}{3\gamma}\right)\Psi_{10}Y_{01}\mathcal{N}^{\frac{1}{3}-\frac{2}{3\gamma}}\xi^{-\frac{2}{3}+\frac{1}{3\gamma}} + O\left(\xi^{-1+\frac{2}{3\gamma}}\right)\right]$$

$$u = 1 + U_{00}\mathcal{N}^{-\frac{2}{3\gamma}}\xi^{-\frac{1}{2}+\frac{2}{3\gamma}} + \left(\frac{2}{3\gamma}\Psi_{10}U_{00} + U_{10}\right)\mathcal{N}^{-\frac{2}{3}-\frac{2}{3\gamma}}\xi^{-\frac{2}{3}+\frac{1}{3\gamma}} + O\left(\xi^{-1+\frac{2}{3\gamma}}\right)$$

$$v = \xi^{-\frac{1}{4}}\left[V_{00} + \left(\mathcal{N}V_{01} + V_{10}\right)\mathcal{N}^{-\frac{2}{3\gamma}}\xi^{-\frac{1}{2}+\frac{2}{3\gamma}} + \left\{\left(-1+\frac{2}{3\gamma}\right)V_{01}\mathcal{N} + \right.\right. \qquad (10.4)$$

$$\left.\left. + \frac{2}{3\gamma}V_{10}\right\}\Psi_{10}\mathcal{N}^{-\frac{2}{3}-\frac{2}{3\gamma}}\xi^{-\frac{2}{3}+\frac{1}{3\gamma}} + O\left(\xi^{-1+\frac{2}{3\gamma}}\right)\right]$$

$$p = \xi^{-\frac{1}{2}}\left[P_{00} + O\left(\xi^{-\frac{1}{2}}\right)\right]$$

$$\rho = R_{00}\mathcal{N}^{\frac{2}{3\gamma}}\xi^{-\frac{1}{3\gamma}} + \left(-\frac{2}{3\gamma}\Psi_{10}R_{00} + R_{10}\right)\mathcal{N}^{-\frac{2}{3}+\frac{2}{3\gamma}}\xi^{-\frac{1}{6}-\frac{1}{3\gamma}} + O\left(\xi^{-\frac{1}{2}}\right) \qquad (10.4)$$

$$h = H_{00}\mathcal{N}^{-\frac{2}{3\gamma}}\xi^{-\frac{1}{2}+\frac{1}{3\gamma}} + \left(\frac{2}{3\gamma}\Psi_{10}H_{00} + H_{10}\right)\mathcal{N}^{-\frac{2}{3}-\frac{2}{3\gamma}}\xi^{-\frac{2}{3}+\frac{1}{3\gamma}} + O\left(\xi^{-1+\frac{2}{3\gamma}}\right)$$

These expressions predict the form of the asymptotic expansion for the sought functions in the internal region of the flow, namely:

$$y=\xi^{\frac{3}{4}}\left[y_0(N)+\xi^{-\frac{1}{2}+\frac{1}{3\gamma}}y_1(N)+\xi^{-\frac{2}{3}+\frac{1}{3\gamma}}y_2(N)+\cdots\right]$$

$$u=u_0(N)+\xi^{-\frac{1}{2}+\frac{1}{3\gamma}}u_1(N)+\xi^{-\frac{2}{3}+\frac{1}{3\gamma}}u_2(N)+\cdots]$$

$$v=\xi^{-\frac{1}{4}}\left[v_0(N)+\xi^{-\frac{1}{2}+\frac{1}{3\gamma}}v_1(N)+\xi^{-\frac{2}{3}+\frac{1}{3\gamma}}v_2(N)+\cdots\right]$$

$$p=\xi^{-\frac{1}{2}}\left[p_0(N)+\xi^{-\frac{1}{2}+\frac{1}{3\gamma}}p_1(N)+\xi^{-\frac{2}{3}+\frac{1}{3\gamma}}p_2(N)+\cdots\right] \qquad (10.5)$$

$$\rho=\xi^{-\frac{1}{2}}\left[\rho_0(N)+\xi^{-\frac{1}{2}+\frac{1}{3\gamma}}\rho_1(N)+\xi^{-\frac{2}{3}+\frac{1}{3\gamma}}\rho_2(N)+\cdots\right]$$

$$h=h_0(N)+\xi^{-\frac{1}{2}+\frac{1}{3\gamma}}h_1(N)+\xi^{-\frac{2}{3}+\frac{1}{3\gamma}}h_2(N)+\cdots$$

Here, the internal and external expansions may be combined, provided that the following boundary conditions for the functions of the internal expansion are satisfied:

$$\left.\begin{aligned} y_0(N)&\to Y_{00}\\ u_0(N)&\to 1\\ p_0(N)&\to P_{00}\\ h_0(N)&\to 0\end{aligned}\right\}\quad \text{at}\quad N\to\infty \qquad (10.6)$$

in the first approximation;

$$\left.\begin{aligned} y_1(N)&\to Y_{01}\,N^{1-\frac{2}{3\gamma}}\\ u_1(N)&\to V_{00}\,N^{-\frac{2}{3\gamma}}\\ p_1(N)&\to 0\\ h_1(N)&\to H_{00}\,N^{-\frac{2}{3\gamma}}\end{aligned}\right\}\quad \text{at}\quad N\to\infty \qquad (10.7)$$

in the second approximation, and

$$\left.\begin{aligned} y_2(\mathcal{N}) &\to -\left(1-\frac{2}{3\gamma}\right)\Psi_{10}Y_{01}\,\mathcal{N}^{-\frac{2}{3\gamma}+\frac{1}{3}} \\ u_2(\mathcal{N}) &\to \left(\frac{2}{3\gamma}\Psi_{10}U_{00}+U_{10}\right)\mathcal{N}^{-\frac{2}{3}-\frac{2}{3\gamma}} \\ p_2(\mathcal{N}) &\to 0 \\ h_2(\mathcal{N}) &\to \left(\frac{2}{3\gamma}\Psi_{10}H_{00}+H_{10}\right)\mathcal{N}^{-\frac{2}{3}-\frac{2}{3\gamma}} \end{aligned}\right\} \quad \text{at } \mathcal{N}\to\infty \qquad (10.8)$$

in the third approximation.

11. Substituting the expansion (10.5) into the initial system of equations (7.4), and equating the principal terms, we obtain the system of equations of the first approximation, which may be written in the form:

$$p_0=\frac{\gamma-1}{\gamma}\rho_0 h_0 = \text{const}$$

$$\frac{\gamma}{\gamma-1}p_0u_0(u_0u_0')'+\frac{1}{4}\mathcal{N}u_0u_0'+\frac{\gamma-1}{2\gamma}h_0=0$$

$$\frac{\gamma}{\gamma-1}p_0\left[u_0\left(\frac{h_0}{\sigma}+\frac{u_0^2}{2}\right)'\right]'+\frac{1}{4}\mathcal{N}\left(h_0+\frac{u_0^2}{2}\right)'=0$$

$$\rho_0u_0y_0'=1 \quad , \quad v_0=u_0\left(\frac{3}{4}y_0-\frac{1}{4}\mathcal{N}y_0'\right) \qquad (11.1)$$

The boundary conditions for these equations are the conditions (10.6) as well as the conditions on the rigid surface which, on the basis of (7.6), (7.7), we write in the form:

$$y_0(0)=u_0(0)=h_0'(0)=0\;; \qquad (11.2)$$

this means, we assume that in the first approximation, the body in the flow is a plane semi-infinite plate. If the Prandtl number $\sigma = 1$, the integral of the heat flux equation, that satisfies the boundary conditions (10.6) and (11.2), is

$$h_0 + \frac{u_0^2}{2} = \frac{1}{2} ; \qquad (11.3)$$

The following will be limited only to this case. The momentum equation reduces then to the form

$$\frac{\gamma}{\gamma-1} p_0 u_0 (u_0 u_0')' + \frac{1}{4} N u_0 u_0' + \frac{1}{4} \frac{\gamma-1}{\gamma} (1 - u_0^2) = 0, \qquad (11.4)$$

where, in correspondence with the third of the boundary conditions (10.6),

$$p_0 = P_{00} \qquad (11.5)$$

The boundary conditions of (11.4) are the second of the conditions (10.6), and (11.2)*).

After determining $u_0(N)$, the function $y_0(N)$ is obtained by integrating the fourth of the equations (11.1), taking (11.3) and (11.2) into account, this yields:

$$y_0 = \frac{\gamma-1}{2\gamma} \frac{1}{p_0} \int_0^N \frac{1-u_0^2}{u_0} \, dN \qquad (11.6)$$

Finally, the first of the boundary conditions (10.6) leads to the relation:

$$Y_{00} = \frac{\gamma-1}{2\gamma} \frac{1}{P_{00}} \int_0^\infty \frac{1-u_0^2}{u_0} \, dN \qquad (11.7)$$

in which the integrand parametrically depends on P_{00}. Hence, (11.7) is the necessary boundary condition for the external first-approximation problem, by means of which the quantities Y_{00} and P_{00} are now related. This means that it uniquely defines the constant C, i.e., the shape of the shock-wave surface, and completely closes the system of relations of the first approximation. The problem of the flow past a semi-infinite plate, thus stated, has been solved in /3/.

12. Let us now examine the second and third approximations. First, on the basis of the second of the equations (7.4) and the boundary condition (10.7), we get

$$p_1 = \frac{\gamma-1}{\gamma}\left(\rho_0 h_1 + h_0 \rho_1\right) = 0 \tag{12.1}$$

The third of the equations (7.4), after certain transformations taking the relations of the first approximation into account, leads to the integral

$$h_1 + u_0 u_1 = 0 \tag{12.2}$$

This solution satisfies the boundary conditions (10.7), since in accordance with (2.8) and (8.7) $H_{00} + U_{00} = 0$. It also satisfies with the required order of approximation the boundary conditions at the wall, which can be readily seen by substituting the expansions (10.5) into (7.7).

Now, after certain transformations, the first of the momentum equations (7.4) leads to the following equation for function u_1:

$$\frac{\gamma}{\gamma-1} p_0 \left(u_0 u_1\right)'' + \frac{1}{4} N u_1' - \left[\frac{\gamma-1}{4\gamma}\,\frac{1+u_0^2}{u_0^2} + \left(-\frac{1}{2} + \frac{1}{3\gamma}\right)\right] u_1 = 0 \tag{12.3}$$

Its boundary conditions are the second of the conditions (10.7) and the condition of attachment (7.7), i.e.,

$$u_1(0) = 0 \quad ; \quad u_1(N) \to U_{00} N^{-\frac{2}{3\gamma}} \quad \text{at} \quad N \to \infty \tag{12.4}$$

Finally, the function $y_1(N)$ satisfies the differential equation

$$y_1' + \frac{\gamma-1}{2\gamma p_0}\,\frac{1+u_0^2}{u_0^2}\,u_1 = 0; \tag{12.4}$$

as a boundary condition for this equation serves the first of the conditions (10.7)*):

Integration of (12.4) yields the value of the function $y_1(0)$ at the wall, that determines its form in the second approximation. In the same manner is found the systems of relations of the third approximation: the integrals

$$p_2 = \frac{\gamma-1}{\gamma}\left(\rho_0 h_2 + h_0 \rho_2\right) = 0 \qquad (12.5)$$

$$h_2 + u_0 u_2 = 0 \qquad (12.6)$$

the differential equation for the function $u_2(N)$:

$$\frac{\gamma}{\gamma-1} p_0 \left(u_0 u_2\right)'' + \frac{1}{4} N u_2' - \left[\frac{\gamma-1}{4\gamma}\,\frac{1+u_0^2}{u_0^2} + \left(-\frac{2}{3} + \frac{1}{3\gamma}\right)\right] u_2 = 0 \qquad (12.7)$$

with the boundary conditions

$$u_2(0) = 0 \ , \quad u_2(N) \to \left(\frac{2}{3\gamma}\Psi_{10} V_{00} + V_{10}\right) N^{-\frac{2}{3}-\frac{2}{3\gamma}} \ \text{при}\ N \to \infty \qquad (12.8)$$

as well as the equation for the function ?

$$y_2' + \frac{\gamma-1}{2\gamma p_0}\,\frac{1+u_0^2}{u_0^2}\, u_2 = 0, \qquad (12.9)$$

the solution of which must satisfy the first of the conditions (10.8). As a result, may be obtained the value of the function $y_2(0)$.

Thus, the sought shape of the wall on which takes place a pressure distribution prescribed in the first-approximation theory to a plane semi-infinite plate, is

$$y \cong y_1(0)\,\xi^{\frac{1}{4}+\frac{1}{3\gamma}} + y_2(0)\,\xi^{\frac{1}{12}+\frac{1}{3\gamma}} \qquad (12.10)$$

Note, that in corespondance with the previously performed estimates of the neglected

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*) Note, that the character of the asymptotic behavior of all functions of the internal expansion, as prescribed by the boundary conditions (10.6) through (10.8), completely corresponds to the character of the behavior resulting from an analysis of the differential equations for these functions.

terms, this result contains a relative error on the order of $\xi^{-1+\frac{2}{3\gamma}}$ or $\xi^{-\frac{1}{2}}$, whereas the first-approximation theory involves a relative error on the order of $\xi^{-\frac{1}{2}+\frac{1}{3\gamma}}$.

13. Synthesizing the results, we write the final expressions for the pressure distribution $\bar{p}$ over the surface of the body in the flow, and the shape of the body. To this end, as usual, we consider the Reynolds number of the problem

$$Re_\infty = \frac{\rho_\infty V_\infty \bar{x}}{\mu_\infty} \qquad (13.1)$$

where $\bar{x}$ is the dimensional distance from the leading edge.

The resulting expression for the pressure at the body is then:

$$\frac{\bar{p}}{p_\infty} \cong \gamma\sqrt{\gamma-1}\, P_0 \frac{M_\infty^3}{\sqrt{Re_\infty}} \qquad (13.2)$$

and for its relative thickness:

$$\frac{\bar{y}}{\bar{x}} \cong y_1(0)\left(\frac{\sqrt{\gamma-1}\, M_\infty}{\sqrt{Re_\infty}}\right)^{\frac{3}{2}-\frac{2}{3\gamma}} + y_2(0)\left(\frac{\sqrt{\gamma-1}\, M_\infty}{\sqrt{Re_\infty}}\right)^{\frac{11}{6}-\frac{2}{3\gamma}} \qquad (13.3)$$

The formula (13.3) defines the asymptotic shape of the contour of the body on which takes place the pressure distribution (13.2).

The results of numerical calculations performed for the case $\gamma = 1.4$, give the following values for the sought constants:

$$y_1(0) = 0.7460, \qquad y_2(0) = 2.2752.$$

CONCLUSION

The performed investigation demonstrates that by treating the problem of the hypersonic flow of a gas at $M_{\infty} \to \infty$ past a slender body as a problem of the strong interaction of the boundary layer at the surface of the body with the inviscid region of the flow field, it is possible to obtain a solution of this problem at a higher order of approximation than by the techniques priviously used. Further improvement of the accuracy of the obtained results (determination of the higher-order terms of asymptotic expansions) will lead to the necessity of accounting for the viscosity in the external portion of the flow field, and to additional terms in the equations (that are neglected in boundary layer theory) for the internal region. As has been shown in the works /1, 5/, however, such a treatment, strictly speaking, is inadmissible because the terms that have to be considered in the Navier Stokes equations are of the same order as the Barnett terms which these equations to not take into account.

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